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Lecture 10: Sequences and Summations (2)

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Recap Previous Lecture

- *A **sequence** is an ordered* list of elements: f(n) = a_n.
- Geometric sequence: a, ar, ar^2, ar^3, …, ar^{n-1}
- Arithmetic sequence: a, a+d, a+2d, a+3d, …, a+(n-1)d
- Summations

$$S = \begin{cases} a(1-r^n)/(1-r) & \text{if } r \neq 1 \\ na & \text{if } r = 1 \end{cases}$$

$$S = \frac{n(a_1 + a_n)}{2}$$

$$S = \frac{n(a_1 + a_1 + (n-1)d)}{2}$$

$$S = \frac{n(2a_1 + (n-1)d)}{2}$$

Outline

- Special sequences
- Sum of the elements of a sequence

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- **Special sequences**
- Sum of the elements of a sequence

Sequence Formula

a) 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, …
- The sequence alternates 1's and 0's, increasing the number of 1's and 0's each time

b) 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, …
- This sequence increases by one, but repeats all even numbers once

c) 1, 0, 2, 0, 4, 0, 8, 0, 16, 0, …
- The non-0 numbers are a geometric sequence (2^n) interspersed with zeros

d) 3, 6, 12, 24, 48, 96, 192, …
- Each term is twice the previous: geometric progression
- $a_n = 3*2^{n-1}$

Sequence Formula

e) 15, 8, 1, -6, -13, -20, -27, …
- Each term is 7 less than the previous term
- $a_n = 22 - 7n$

f) 3, 5, 8, 12, 17, 23, 30, 38, 47, …
- The difference between successive terms increases by on each time
- $a_1 = 3$, $a_n = a_{n-1} + n$
- $a_n = n(n+1)/2 + 2$

g) 2, 16, 54, 128, 250, 432, 686, …
- Each term is twice the cube of n
- $a_n = 2*n^3$

h) 2, 3, 7, 25, 121, 721, 5041, 40321
- Each successive term is about n times the previous
- $a_n = n! + 1$

Some useful sequences

- $n^2 = 1, 4, 9, 16, 25, 36, \ldots$
- $n^3 = 1, 8, 27, 64, 125, 216, \ldots$
- $n^4 = 1, 16, 81, 256, 625, 1296, \ldots$
- $2^n = 2, 4, 8, 16, 32, 64, \ldots$
- $3^n = 3, 9, 27, 81, 243, 729, \ldots$
- $n! = 1, 2, 6, 24, 120, 720, \ldots$

Harmonic Sequence

- The sequence: $\{h_n\}_{n=1}^{\infty}$ = 1/n
  is known as the **harmonic** sequence
- The sequence is simply:

  1, 1/2, 1/3, 1/4, 1/5, …
- This sequence is particularly interesting because its summation is divergent:

  $\sum_{n=1}^{\infty}$ (1/n) = ∞

Sequences: Example 1

- Consider the sequence

$$\{(1 + 1/n)^n\}_{n=1}^{\infty}$$

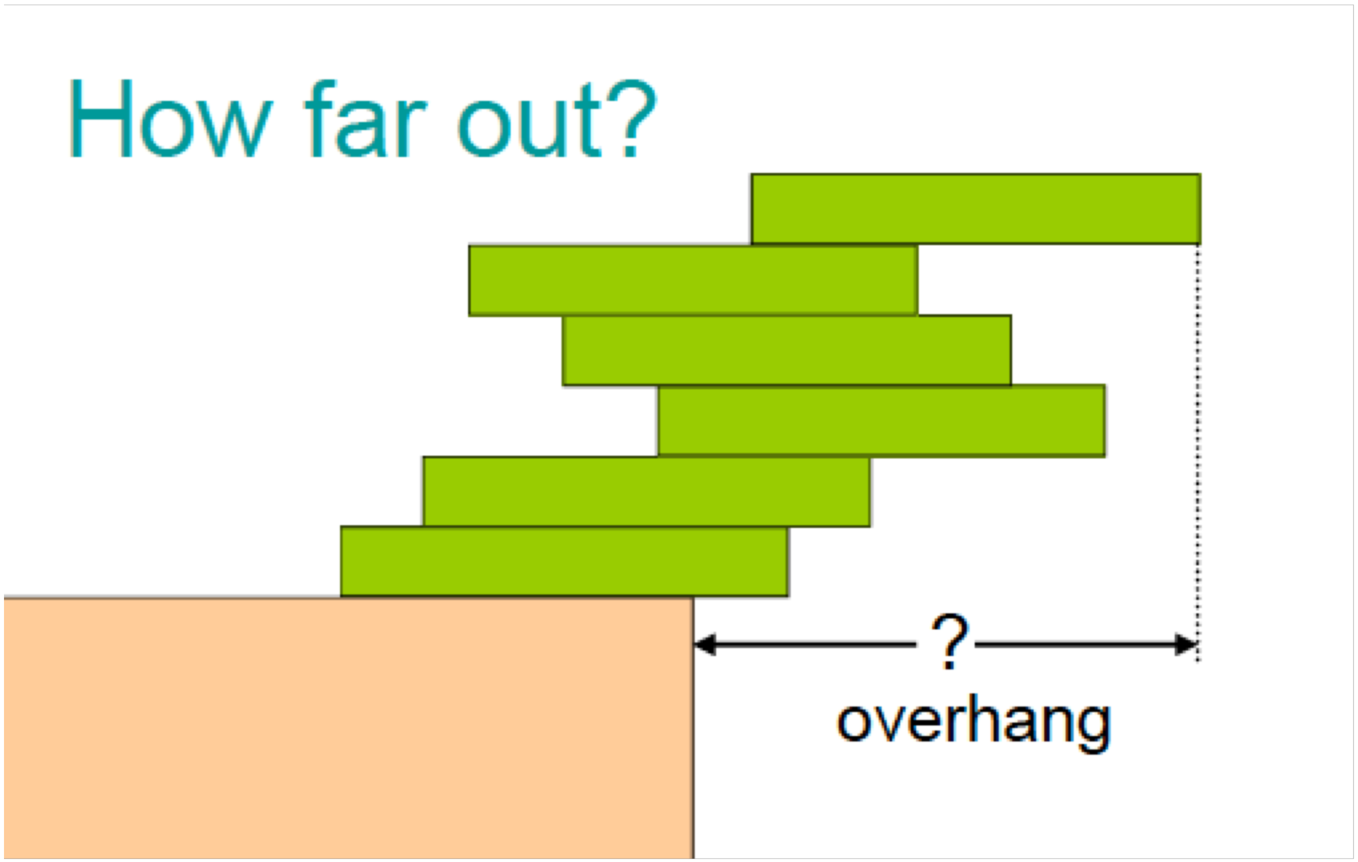
- The terms of the sequence are:

$$a_1 = (1 + 1/1)^1 = 2.00000$$
$$a_2 = (1 + 1/2)^2 = 2.25000$$
$$a_3 = (1 + 1/3)^3 = 2.37037$$
$$a_4 = (1 + 1/4)^4 = 2.44140$$
$$a_5 = (1 + 1/5)^5 = 2.48832$$

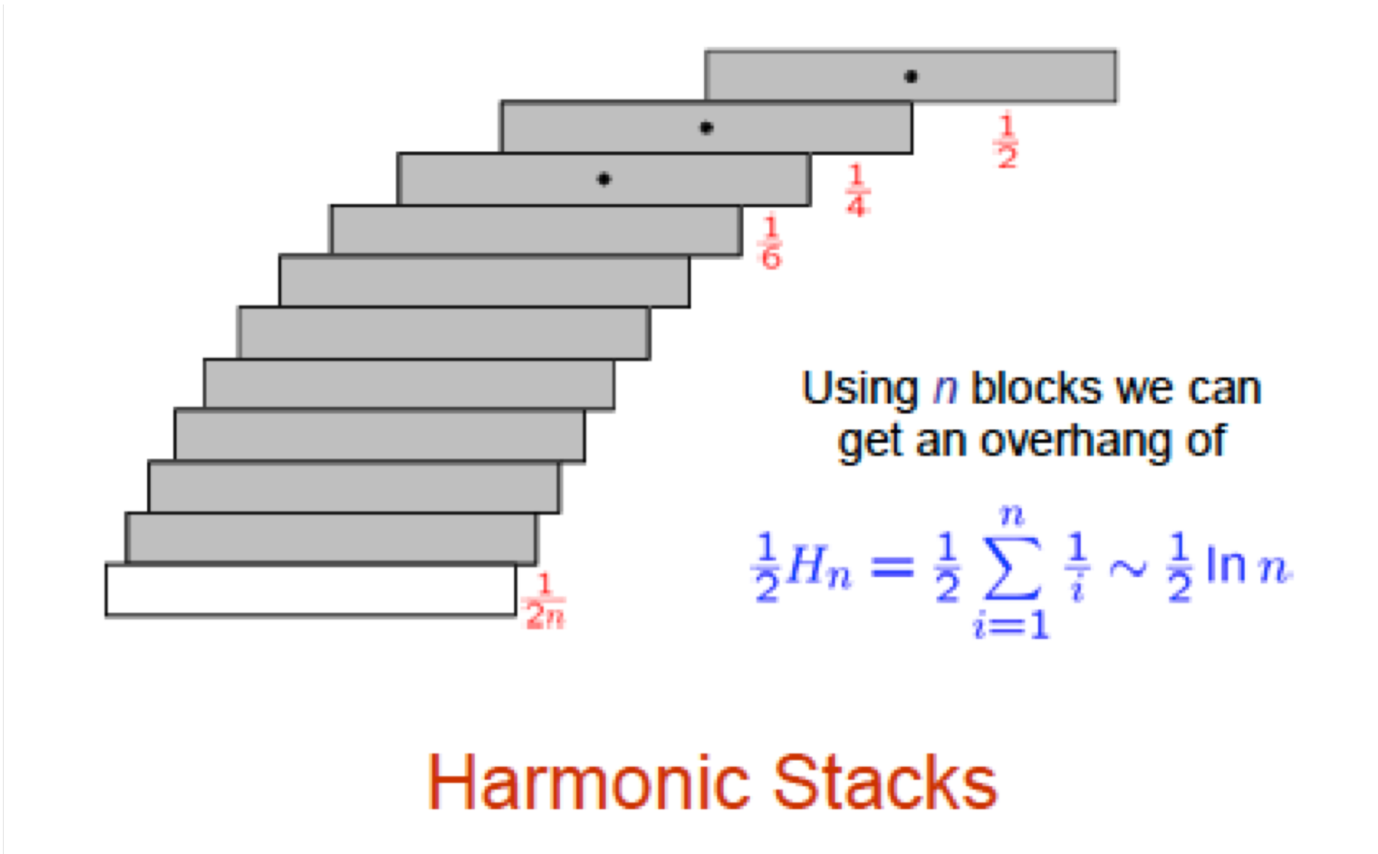
- What is this sequence?

- The sequence corresponds to $\lim_{n\to\infty}\{(1 + 1/n)^n\}_{n=1}^{\infty} = e =$ 2.71828..

Book stacking example



Book stacking example



Outline

- Special sequences
- **Sum of the elements of a sequence**

Summation

- You should be by now familiar with the summation notation:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_{n-1} + a_n$$

  Here
  - i is the index of the summation
  - m is the lower limit
  - n is the upper limit
- Often times, it is useful to change the lower/upper limits, which can be done in a straightforward manner (although we must be very careful):

$$\sum_{i=1}^{n} a_i = \sum_{i=0}^{n-1} a_{i+1}$$

Summation

- A summation:

upper limit

$\sum_{j=m}^{n} a_j$ or $\sum_{j=m}^{n} a_j$

lower limit

index of summation

- is like a for loop:

```
int sum = 0;
for ( int j = m; j <= n; j++ )
      sum += a(j);
```

Evaluating sequences

$\sum_{k=1}^{5}(k+1)$

- $2 + 3 + 4 + 5 + 6 = 20$

$\sum_{j=0}^{4}(-2)^j$

- $(-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4 = 11$

$\sum_{i=1}^{10} 3$

- $3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 30$

$\sum_{j=0}^{8}\left(2^{j+1} - 2^j\right)$

- $(2^1-2^0) + (2^2-2^1) + (2^3-2^2) + \ldots (2^{10}-2^9) = 511$
  - Note that each term (except the first and last) is cancelled by another term

Series

- When we take the sum of a sequence, we get a series
- We have already seen a closed form for geometric series
- Some other useful closed forms include the following:
  - $\sum_{i=k}^{u} 1$ = u-k+1, for k≤u
  - $\sum_{i=k}^{u} i$ = n(n+1)/2
  - $\sum_{i=1}^{n} i^2$ = n(n+1)(2n+1)/6
  - $\sum_{i=1}^{n} i^k \approx$ $n^{k+1}/(k+1)$

Infinite Series

- Although we will mostly deal with finite series (i.e., an upper limit of n for fixed integer), inifinite series are also useful
- Consider the following geometric series:
  - $\Sigma_{n=0}(1/2^n) = 1 + 1/2 + 1/4 + 1/8 + \ldots$ converges to 2
  - $\Sigma_{n=0}(2^n) = 1 + 2 + 4 + 8 + \ldots$ does not converge
- However note: $\Sigma_{n=0} (2^n) = 2^{n+1} – 1$ (a=1, q=2)

Evaluating sequences

- Let S = { 1, 3, 5, 7 }

- What is $\sum_{j \in S} j$
  - $1 + 3 + 5 + 7 = 16$
- What is $\sum_{j \in S} j^2$
  - $1^2 + 3^2 + 5^2 + 7^2 = 84$
- What is $\sum_{j \in S} (1/j)$
  - $1/1 + 1/3 + 1/5 + 1/7 = 176/105$
- What is $\sum_{j \in S} 1$
  - $1 + 1 + 1 + 1 = 4$

Can you evaluate this?

$$\sum_{i=1}^{n} \frac{1}{k(k+1)}$$

Here is the trick. Note that

$$\boxed{\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}}$$

Does it help?

Double Summation

- Like a nested for loop

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij$$

- Is equivalent to:

```
int sum = 0;
for ( int i = 1; i <= 4; i++ )
    for ( int j = 1; j <= 3; j++ )
        sum += i*j;
```

Solve the following

$$1 + 1/2 + 1/4 + 1/8 + \cdots = \sum_{i=0}^{\infty} (1/2)^i$$

$$0.999999999\ldots = 0.9 \sum_{i=0}^{\infty} (1/10)^i$$

$$1 - 1/2 + 1/4 - 1/8 + \cdots = \sum_{i=0}^{\infty} (-1/2)^i$$

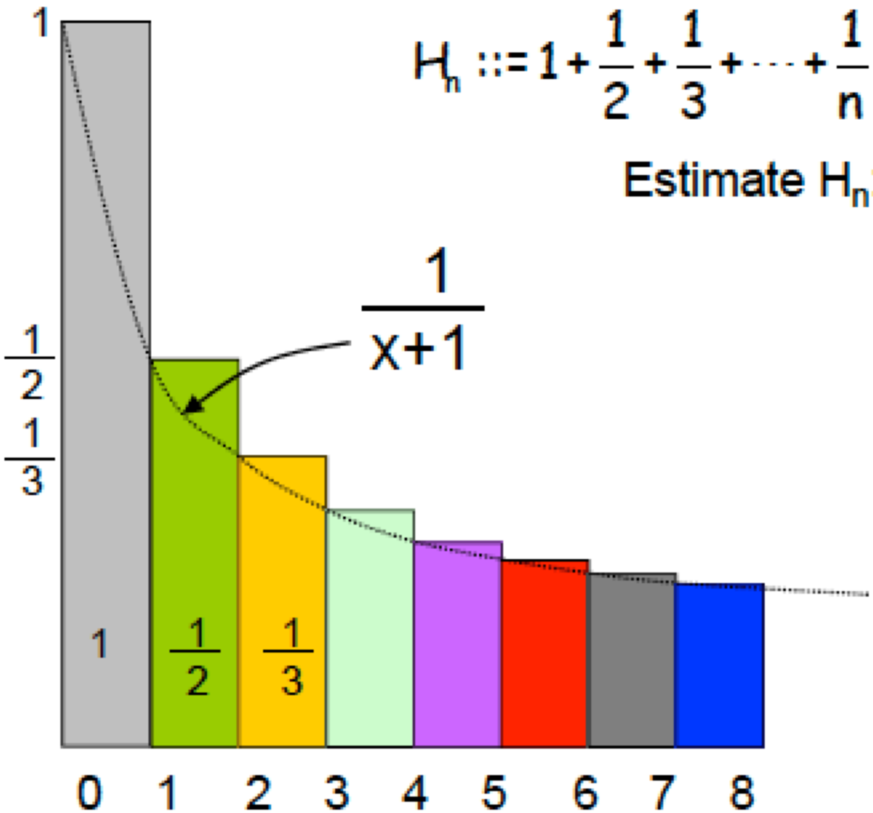
$$1 + 2 + 4 + 8 + \cdots + 2^{n-1} = \sum_{i=0}^{n-1} 2^i$$

$$1 + 3 + 9 + 27 + \cdots + 3^{n-1} = \sum_{i=0}^{n-1} 3^i$$

Sum of harmonic series

$$H_n ::= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Estimate H_n:



$$\int_0^n \frac{1}{x+1}\,dx \le 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

$$\int_1^{n+1} \frac{1}{x}\,dx \le H_n$$

$$\ln(n+1) \le H_n$$

Products

$$\prod_{i=1}^{n} a_i := a_1 \cdot a_2 \cdots a_n$$

$$\prod_{k=1}^{5} k^2$$

$$\prod_{k=1}^{n} \frac{k}{k+1}$$

$$\prod_{k=1}^{n} 2^k$$

Dealing with Products

Factorial defines a product:

$$n! = 1 \cdot 2 \cdot 3 \cdot n = \prod_{i=1}^{n} i$$

How to estimate n!?

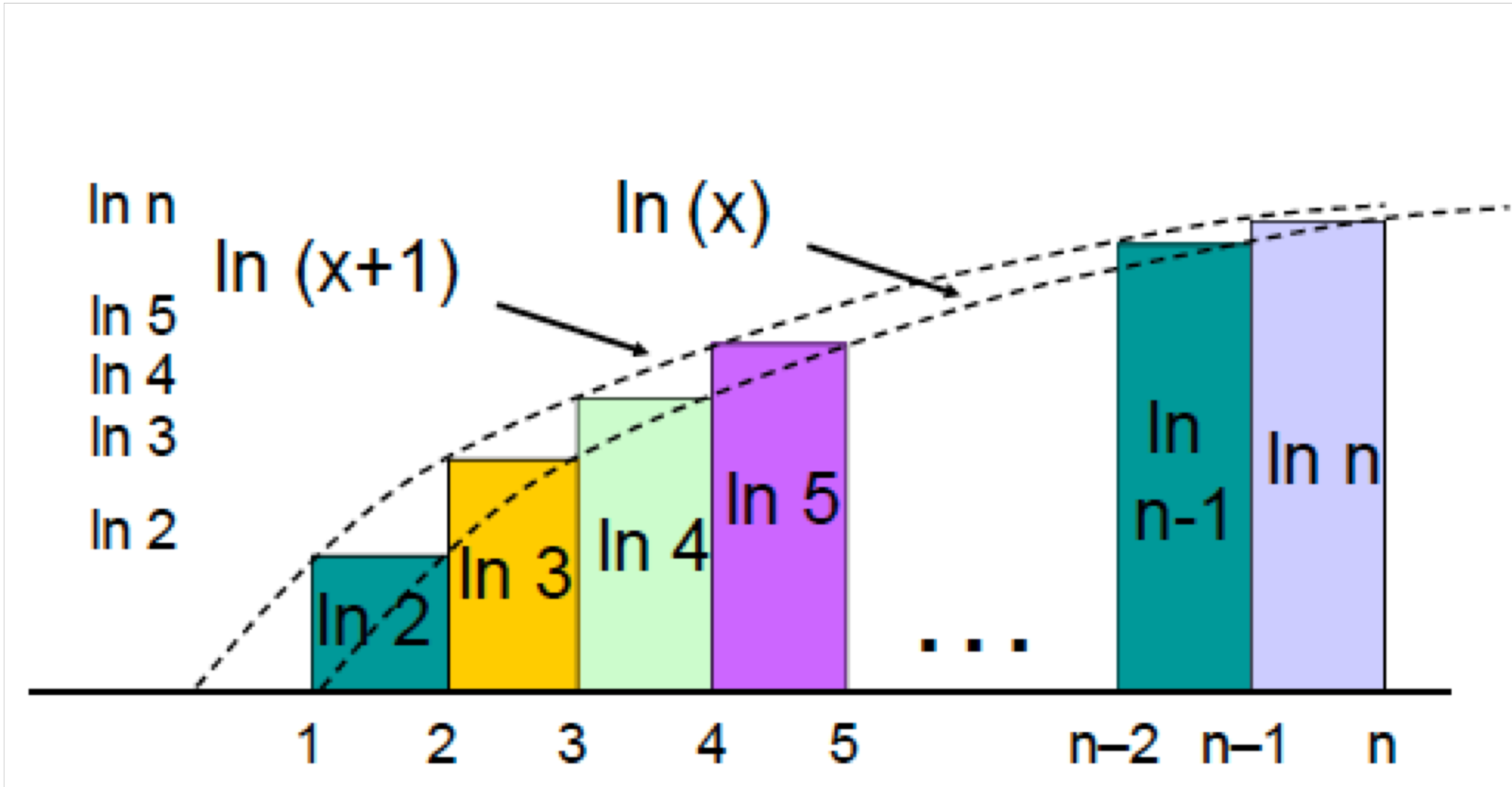
Turn product into a sum taking logs:

$$\ln(n!) = \ln(1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n)$$

$$= \ln 1 + \ln 2 + \cdots + \ln(n-1) + \ln(n)$$

$$= \sum_{i=1}^{n} \ln(i)$$

Factorial



$$\int_1^n \ln(x)\,dx \le \sum_{i=1}^{n} \ln(i) \le \int_0^n \ln(x+1)\,dx$$

Factorial

$$\int_1^n \ln(x)\,dx \le \sum_{i=1}^{n} \ln(i) \le \int_0^n \ln(x+1)\,dx$$

Reminder: $\int \ln x\,dx = x \ln\left(\frac{x}{e}\right)$

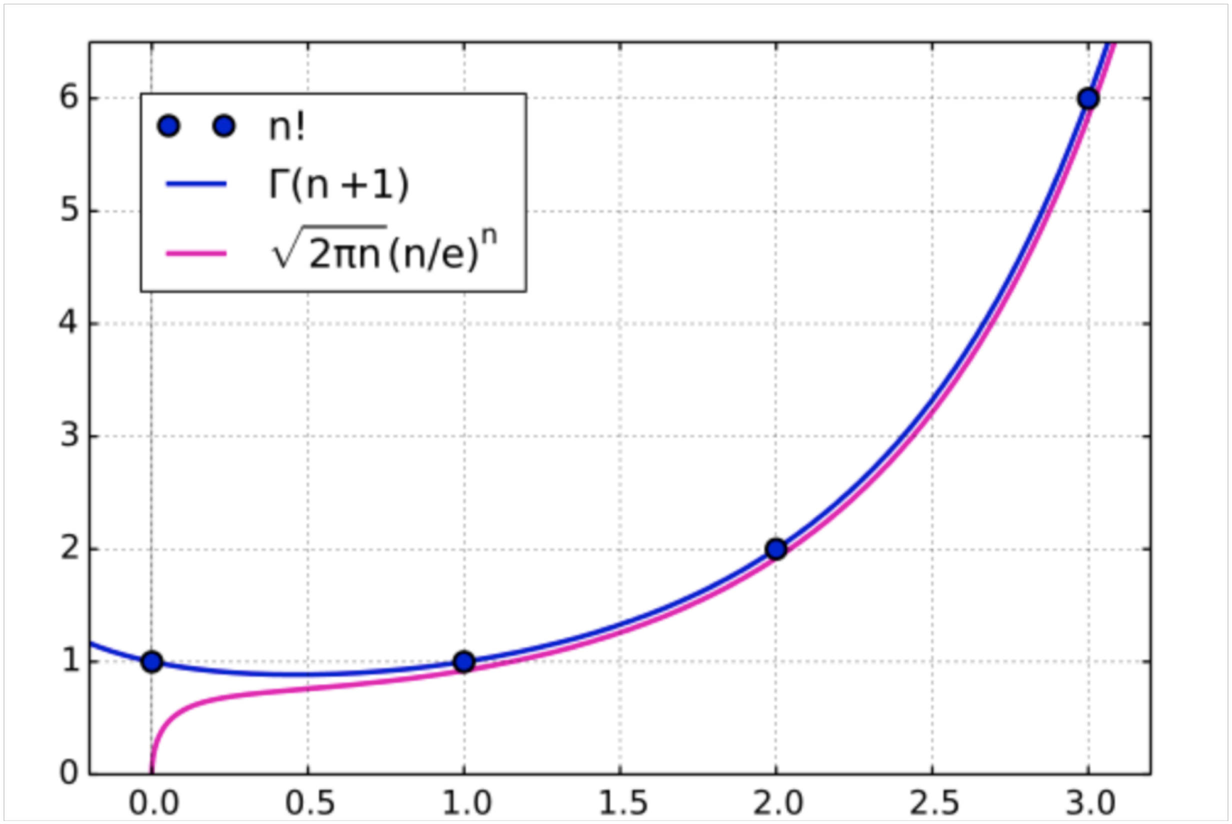
$$n \ln(n/e) + 1 \le \sum \ln(i) \le (n+1)\ln((n+1)/e) + 1$$

so guess: $\sum_{i=1}^{n} \ln(i) \approx \left(n + \frac{1}{2}\right) \ln\left(\frac{n}{e}\right)$

Stirling's formula

$$\sum_{i=1}^{n} \ln(i) \approx \left(n+\frac{1}{2}\right)\ln\left(\frac{n}{e}\right)$$

exponentiating: $n! \approx \sqrt{n/e}\left(\frac{n}{e}\right)^n$



Next class

- Topic: Cardinality of Sets
- Pre-class reading: Chap 2.5