UNIVERSITY AT ALBANY
State University of New York

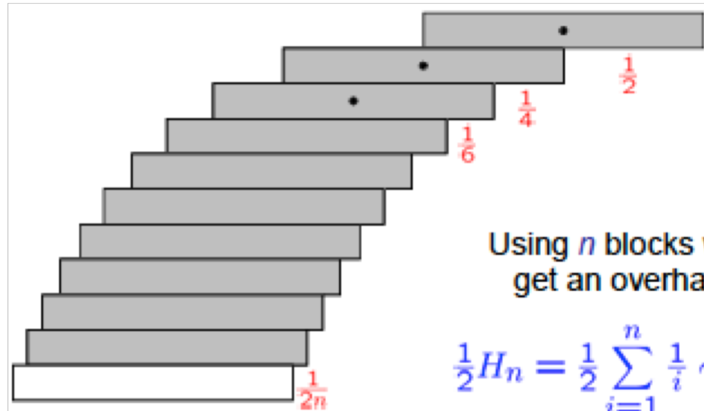
Lecture 11: Cardinality of Sets

Dr. Chengjiang Long
Computer Vision Researcher at Kitware Inc.
Adjunct Professor at SUNY at Albany.
Email: **clong2@albany.edu**

Recap Previous Lecture

- Special Sequences
- Summations

$n^2 = 1, 4, 9, 16, 25, 36, \ldots$
$n^3 = 1, 8, 27, 64, 125, 216, \ldots$
$n^4 = 1, 16, 81, 256, 625, 1296, \ldots$
$2^n = 2, 4, 8, 16, 32, 64, \ldots$
$3^n = 3, 9, 27, 81, 243, 729, \ldots$
$n! = 1, 2, 6, 24, 120, 720, \ldots$



Using n blocks we can get an overhang of

$$\frac{1}{2}H_n = \frac{1}{2}\sum_{i=1}^{n}\frac{1}{i} \sim \frac{1}{2}\ln n$$

- A summation:

$$\sum_{j=m}^{n} a_j \quad \text{or} \quad \sum_{j=m}^{n} a_j$$

upper limit

lower limit

index of summation

- is like a for loop:

```
int sum = 0;
for ( int j = m; j <= n; j++ )
    sum += a(j);
```

20

- Like a nested for loop

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij$$

- Is equivalent to:

```
int sum = 0;
for ( int i = 1; i <= 4; i++ )
    for ( int j = 1; j <= 3; j++ )
        sum += i*j;
```

Solve the following

$$1 + 1/2 + 1/4 + 1/8 + \cdots = \sum_{i=0}^{\infty} (1/2)^i$$

$$0.999999999\ldots = 0.9 \sum_{i=0}^{\infty} (1/10)^i$$

$$1 - 1/2 + 1/4 - 1/8 + \cdots = \sum_{i=0}^{\infty} (-1/2)^i$$

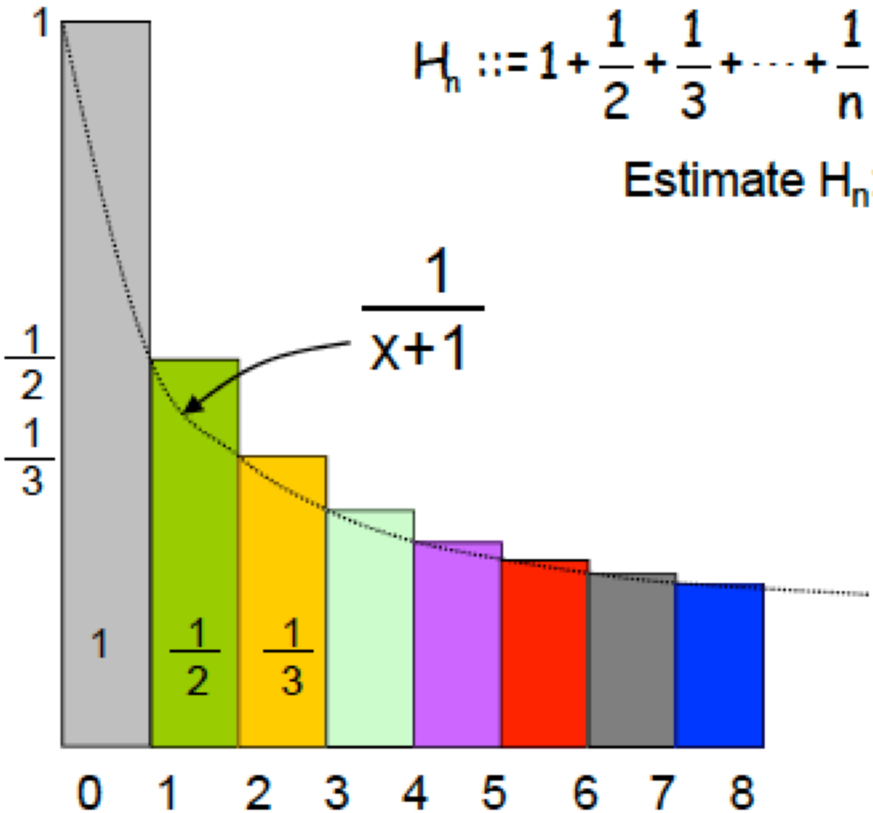
$$1 + 2 + 4 + 8 + \cdots + 2^{n-1} = \sum_{i=0}^{n-1} 2^i$$

$$1 + 3 + 9 + 27 + \cdots + 3^{n-1} = \sum_{i=0}^{n-1} 3^i$$

Sum of harmonic series

$$H_n ::= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Estimate H_n:



$$\int_0^n \frac{1}{x+1}\,dx \le 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

$$\int_1^{n+1} \frac{1}{x}\,dx \le H_n$$

$$\ln(n+1) \le H_n$$

Products

$$\prod_{i=1}^{n} a_i := a_1 \cdot a_2 \cdots a_n$$

$$\prod_{k=1}^{5} k^2$$

$$\prod_{k=1}^{n} \frac{k}{k+1}$$

$$\prod_{k=1}^{n} 2^k$$

Dealing with Products

Factorial defines a product:

$$n! = 1 \cdot 2 \cdot 3 \cdot n = \prod_{i=1}^{n} i$$

How to estimate n!?

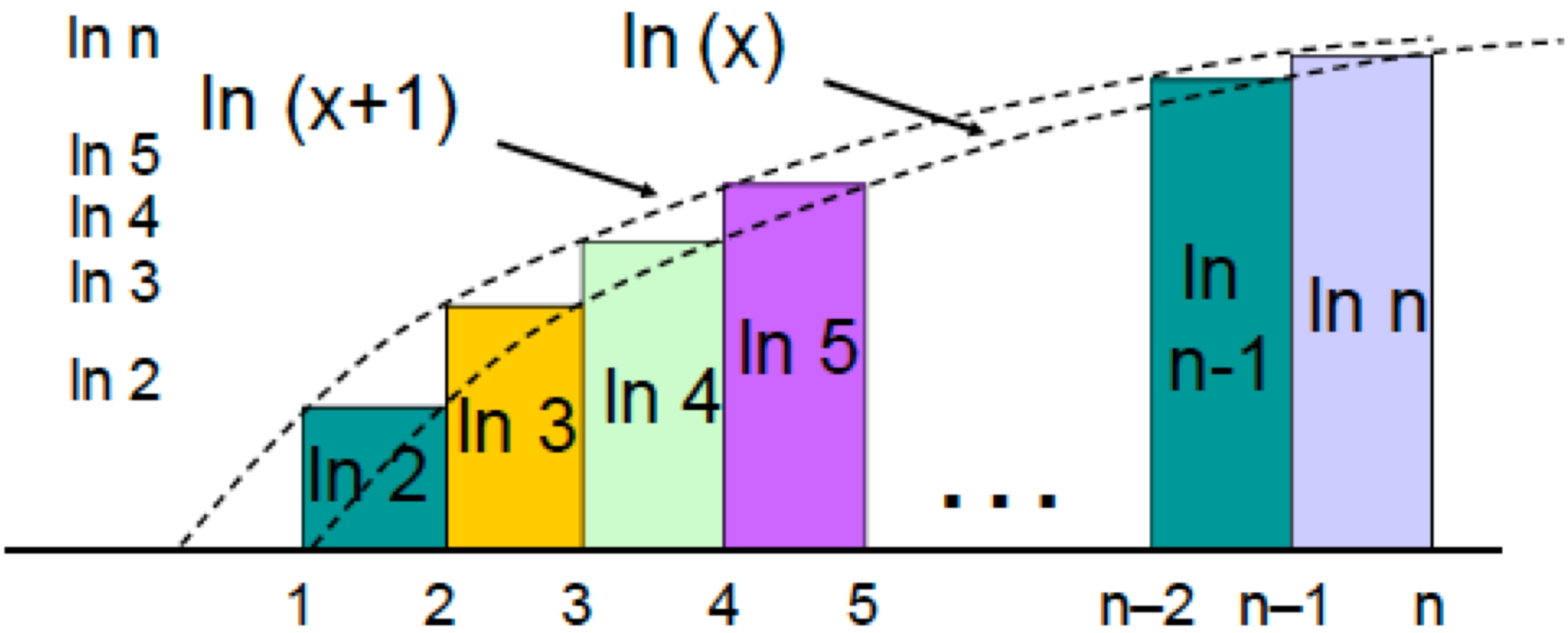
Turn product into a sum taking logs:

$$\ln(n!) = \ln(1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n)$$

$$= \ln 1 + \ln 2 + \cdots + \ln(n-1) + \ln(n)$$

$$= \sum_{i=1}^{n} \ln(i)$$

Factorial



$$\int_1^n \ln(x)\,dx \le \sum_{i=1}^{n} \ln(i) \le \int_0^n \ln(x+1)\,dx$$

Factorial

$$\int_1^n \ln(x)\,dx \le \sum_{i=1}^n \ln(i) \le \int_0^n \ln\,(x+1)dx$$

Reminder: $\int \ln x \, dx = x \ln\left(\frac{x}{e}\right)$

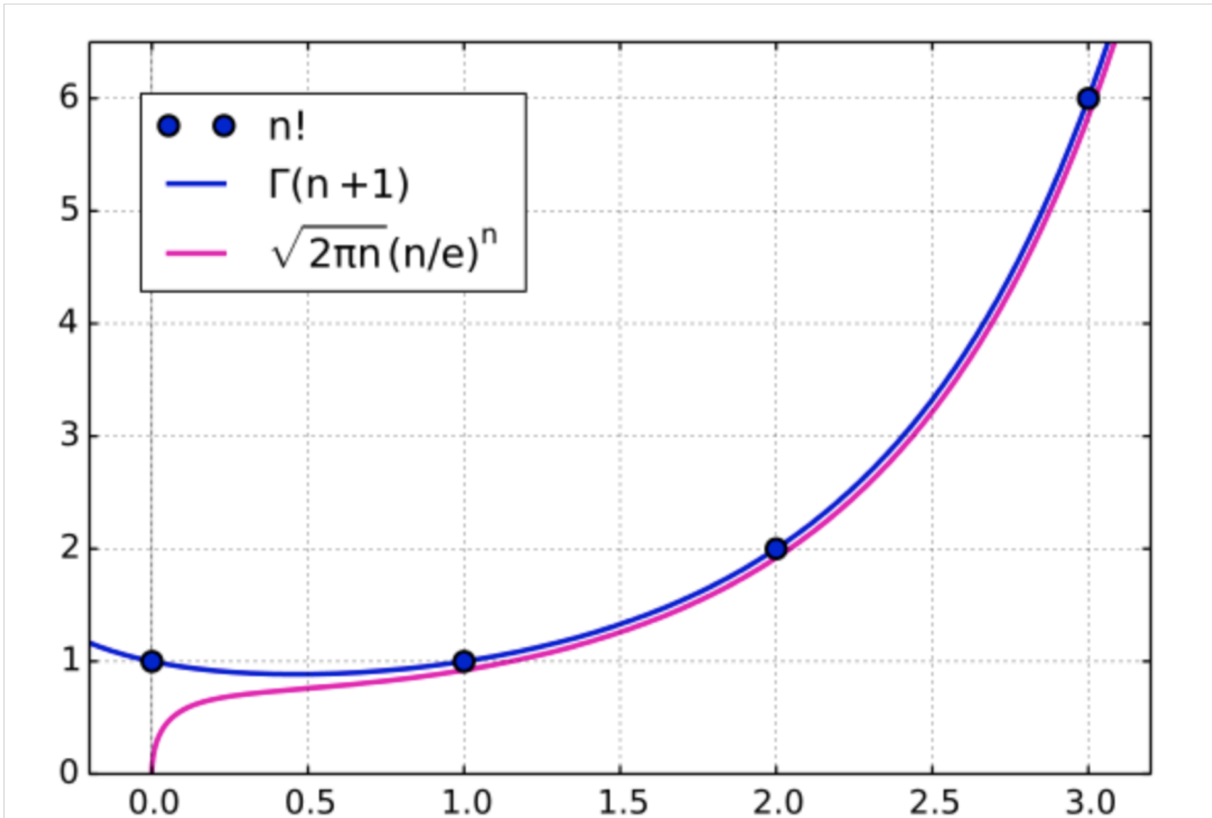
$$n \ln(n/e) + 1 \le \sum \ln(i) \le (n+1) \ln((n+1)/e) + 1$$

so guess: $\sum_{i=1}^{n} \ln(i) \approx \left(n + \frac{1}{2}\right) \ln\left(\frac{n}{e}\right)$

Stirling’s formula

$$\sum_{i=1}^{n} \ln(i) \approx \left(n+\frac{1}{2}\right)\ln\left(\frac{n}{e}\right)$$

exponentiating: $$n! \approx \sqrt{n/e}\left(\frac{n}{e}\right)^{n}$$



Outline

- Introduction to Cardinality
- Equal Cardinality
- Countable Sets and Uncountable Sets
- Cardinality in Practice

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Students in Class

- There is a number of students (set A).
- There is a number of seats in the room (set B).
- How to learn whether the number of students match the number of seats if there is no way to count any or both of students and seats?
- **Answer:** have students to take the seats.

Cardinality

- "The number of elements in a set."
- Let A be a set.

a. If $A = \varnothing$ (the empty set), then the cardinality of A is 0.

b. If A has exactly n elements, n a natural number, then the cardinality of A is n. The set A is a ***finite*** *set*

c. Otherwise, A is an ***infinite*** *set.*

Cardinality Notations

- The cardinality of a set A is denoted by $|A|$.

a. If $A = \varnothing$, then $|A| = 0$.

b. If A has exactly n elements, then $|A| = n$.

c. If A is an infinite set, then $|A| = \infty$.

Examples

- $A = \{2, 3, 5, 7, 11, 13, 17, 19\}$; $|A| = 8$
- $A = N$ (natural numbers); $|N| = \infty$
- $A = Q$ (rational numbers); $|Q| = \infty$
- $A = \{2n \mid n$ is an integer$\}$; $|A| = \infty$

  (the set of even integers)

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Equal Cardinality

DEFINITION:

- Let A and B be sets. Then, $|A| = |B|$ if and only if there is a one-to-one correspondence between the elements of A and the elements of B.

- If there is a one-to-one function (*i.e.*, an **injection**) from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$.

- When $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and write $|A| < |B|$.

Example

1. $A = \{1, 2, 3, 4, 5\}$

   $B = \{a, e, i, o, u\}$

   $1 \rightarrow a,\ 2 \rightarrow e,\ 3 \rightarrow i,\ 4 \rightarrow o,\ 5 \rightarrow u; \quad |B| = 5$

Example

- 2. $A = N$ (the natural numbers)

  $B = \{2n \mid n$ is a natural number$\}$ (the even natural numbers)

  $n \rightarrow 2n$ is a one-to one correspondence between A and B. Therefore, $|A| = |B|$; $|B| = \infty$.

Example

- 3. $A = N$ (the natural numbers)

  $C = \{2n - 1 \mid n$ is a natural number$\}$ (the odd natural numbers)

  $n \rightarrow 2n - 1$ is a one-to one correspondence between A and C. Therefore, $|A| = |C|$; $|C| = \infty$.

Answer the Questions

Which of the following pairs of sets have equal cardinalities:

a) natural numbers and sequence of perfect squares;
b) positive real numbers and negative real numbers;
c) real numbers from the two intervals (0,1) and (1,∞)?

a) If $n \in \mathbb{N},\ n \to n^2$ is a bijection.

b) If $r \in \mathbb{R}, r \to -r$ is a bijection.

c) If $r \in \mathbb{R}, r \to \frac{1}{r}$ is a bijection.

21

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Countable Sets

DEFINITIONS:

- 1. A set S is *finite* if there is a one-to-one correspondence between it and the set $\{1, 2, 3, . . ., n\}$ for some natural number n.
- 2. A set S is *countably infinite* if there is a one-to-one correspondence between it and the natural numbers N.
- *3.* A set S is *countable* if it is either finite or countably infinite.
- 4. A set S is *uncountable* if it is not countable.

Levels of the Infinity

- A set that is either finite or has the **same cardinality** as the set of **natural numbers** (or $\mathbf{Z}^+$) is called **countable**.
- A set that is not countable is **uncountable**.
- The set of real numbers **R** is an uncountable set.
- When an infinite set is countable (**countably infinite**) its cardinality is $\aleph_0$ or "aleph null"(where $\aleph$ is aleph, the 1st letter of the Hebrew alphabet).

Showing That a Set is Countable

- An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).
- The reason for this is that a one-to-one correspondence ***f*** from the set of positive integers to a set S can be expressed in terms of a sequence
    - $a_1, a_2, \ldots, a_n, \ldots$
  - where $a_1 = f(1)$, $a_2 = f(2), \ldots$, $a_n = f(n), \ldots$

Example

- 1. $A = \{1, 2, 3, 4, 5, 6, 7\}$,
  $\Omega = \{a, b, c, d, \ldots x, y, z\}$ are finite sets;
  $|A| = 7, \ |\Omega| = 26$.

- 2. N (the natural numbers), Z (the integers), and Q (the rational numbers) are countably infinite sets; that is,
  $|Q| = |Z| = |N|$.

- 3. I (the irrational numbers) and $\Re$ (the real numbers) are uncountable sets; that is
  $|I| > |N|$ and $|\Re| > |N|$.

Some Facts

1. A set S is finite if and only if for any proper subset $A \subset S$, $|A| < |S|$; that is, "proper subsets of a finite set have fewer elements."

2. Suppose that A and B are infinite sets and $A \subseteq B$. If B is countably infinite then A is countably infinite and $|A| = |B|$.

3. Every subset of a countable set is countable.

4. If A and B are countable sets, then $A \cup B$ is a countable set.

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Practice

- **Problem:** Prove that Cartesian product of two countable sets is a countable set.
- **Solution:**
- Let A and B are countable sets. Then $A \times B$ is a set of ordered pairs $\langle a, b \rangle$ such that $a \in A$ and $b \in B$.
- If we group all pairs that have the same first element ($\forall a \in A \rightarrow \{a\} \times B$) then there is a bijective function for each group from B to the group. As B is countable set then each group is a countable set too.
- Number of such groups is equal to the number of elements of A, which is countable set. Hence countable sequence of the countable sets is a countable set, as desired.
- **At home:** Prove that **N** x **N** is countable.

Practice

- **Problem:** Show that the set of finite strings (words) *W* over an alphabet $A = \{0, 1\}$ is countably infinite.
- **Solution**:

1. It is easy to define a bijective function from **N** to *A*.
2. Let's group all words by the lengths starting from 1 letter.
3. All words in a group could be ordered in "a dictionary" order.
4. Ordering implies a bijection from **N** to a set, *i.e.* makes a set countable.
5. Countable sequence of countable sets is a countable set.

At home: Show that the set of finite strings (words) *W* over any finite alphabet *A* is countably infinite.

If the Set of C programs Countable?

- **Problem**: Show that the set of all C programs is countable.

Solution: Let's consider a C code as a one string constructed from the characters which can appear in a C program. Then see the previous slides.

Cardinality in Use

- The idea of the cardinality of sets is used to compare finite, countable infinite and uncountable infinite sets.
- One of the most important theorems of the theory of sets says that the set and the power set of this set may not have equal cardinality.
- The theory may result in paradoxes in practice:
  - A barber follows the rule to shave everybody in the town who does not shave himself.
  - Should he shave himself?
- Suppose x is a set of the sets that are not elements of itself. Would be x an element of x:
  - If $\forall y \in x \Leftrightarrow x \notin x$, then $\exists\ y = x$, such that $x \in x \Leftrightarrow x \notin x$.
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Next class

- Topic: Matrices
- Pre-class reading: Chap 2.6