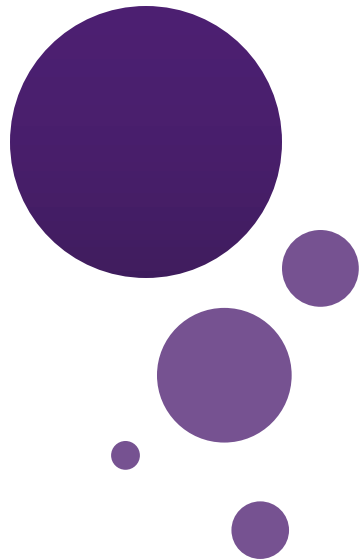




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Lecture 30: Conditional Probabilities

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About the Midterm Exam 2

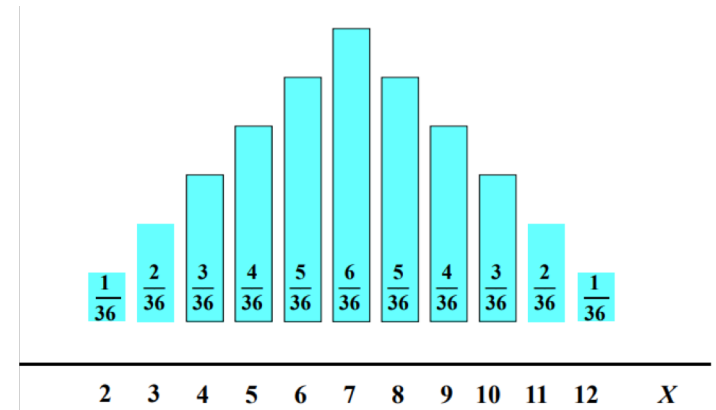
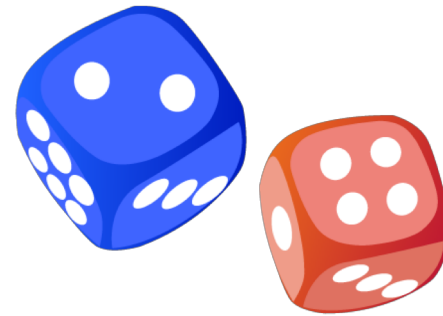
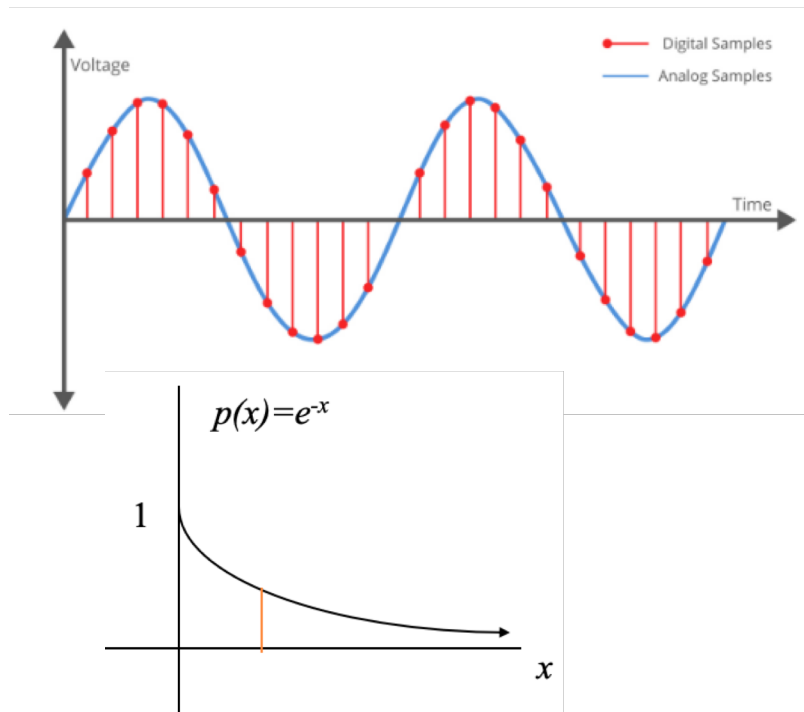
- The 2nd Midterm Exam will be taken on April 17, 2019 at the LC 25 from 11:30 AM to 12:25 PM.
- It will cover Chap 4 to Chap 6.
- Note that this is a close book and close note exam.
- The format is the similar to the Midterm Exam 1.

Problem	Points	Scores
Problem 1: Greatest Common Divisor and Least Common Multiple	25	
Problem 2: Integer Representations and Modular Exponentiation	20	
Problem 3: Induction and Recursion	20	
Problem 4: Counting	35	

(Will be used in Midterm Exam 2)

Recap Previous Lecture

- Random Variable
- Probability Functions



Outline

- Random variables and statistical independence
- Conditional Probability
- Sum and Product Rules
- Bayes Rule

Outline

- **Random variables and statistical independence**
- Conditional Probability
- Sum and Product Rules
- Bayes Rule

Pairs of Discrete Random Variables

- Let x and y be two discrete r.v.
- For each possible pair of values, we can define a joint probability $P(\mathbf{x}, \mathbf{y})$
- We can also define a joint probability mass function $P(\mathbf{x}, \mathbf{y})$ which offers a complete characterization of the pair of

$$P_x(x) = \sum_{y \in Y} P(x, y)$$

$$P_y(y) = \sum_{x \in X} P(x, y)$$

Marginal distributions

Statistical Independence

- Two random variables x and y are said to be independent, if and only if

$$P(x,y)=P_x(x) P_y(y)$$

that is, when knowing the value of x does not give us additional information for the value of y .

- Or, equivalently

$$E[f(x)g(y)] = E[f(x)] E[g(y)]$$

for any functions $f(x)$ and $g(y)$.

Outline

- Random variables and statistical independence
- **Conditional Probability**
- Sum and Product Rules
- Bayes Rule

Conditional Probability

- When two r.v. are not independent, knowing one allows better estimate of the other (e.g. outside temperature, season)

$$\Pr[x = x_i | y = y_j] = \frac{\Pr[x = x_i, y = y_j]}{\Pr[y = y_j]}$$

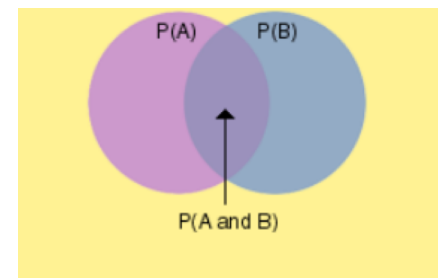
- If independent $P(x|y)=P(x)$

Conditional Probability Example

- A jar contains black and white marbles.
 - Two marbles are chosen without replacement.
 - The probability of selecting a black marble and then a white marble is 0.34.
 - The probability of selecting a black marble on the first draw is 0.47.
- What is the probability of selecting a white marble on the second draw, given that the first marble draw was black?

$$P(\text{White} \mid \text{Black}) = \frac{P(\text{Black} \wedge \text{White})}{P(\text{Black})} = \frac{0.34}{0.47} = 0.72$$

A is black in first draw, B is white in second draw

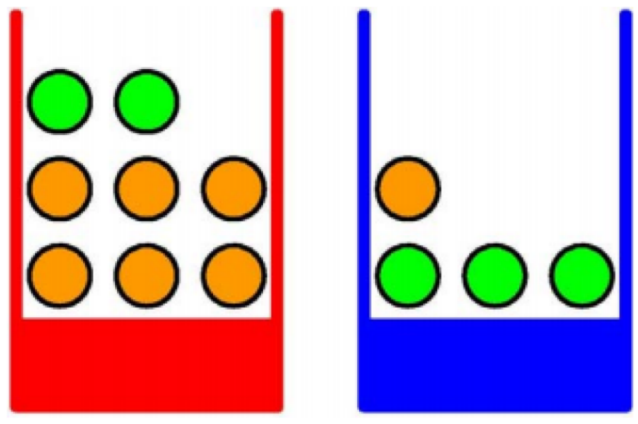


Outline

- Random variables and statistical independence
- Conditional Probability
- **Sum and Product Rules**
- Bayes Rule

Sum and Product Rules

- Example:
 - We have two boxes: one red and one blue
 - Red box: 2 apples and 6 oranges
 - Blue box: 3 apples and 1 orange



[C.M. Bishop, *“Pattern Recognition and Machine Learning”*, 2006]

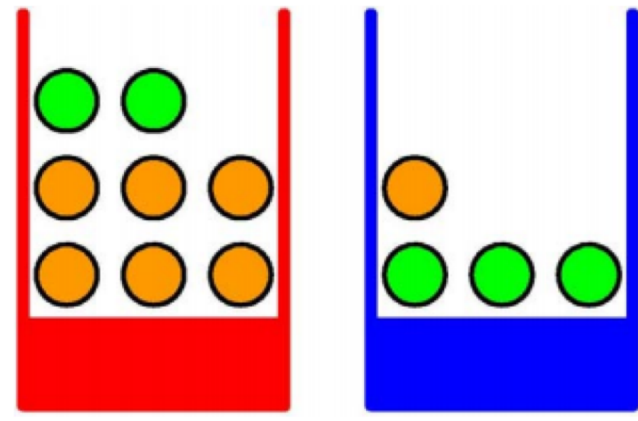
Sum and Product Rules

□ Define:

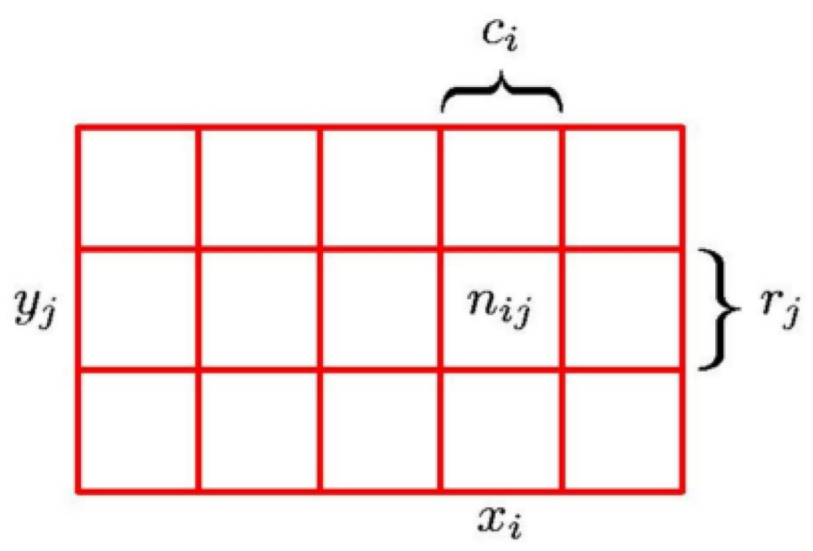
- B random variable for box picked (r or b)
- F identity of fruit (a or o)

□ $p(B=r)=8/12$ and $p(B=b)=4/12$

- Events are mutually exclusive and include all possible outcomes $\Rightarrow$ their probabilities must sum to 1.



Sum and Product Rules



Joint Probability

$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{N}$$

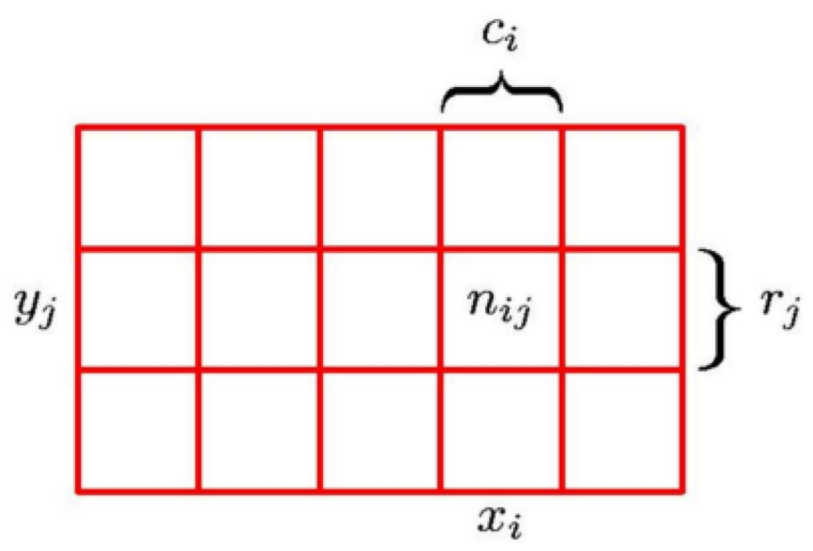
Marginal Probability

$$p(X = x_i) = \frac{c_i}{N}.$$

Conditional Probability

$$p(Y = y_j | X = x_i) = \frac{n_{ij}}{c_i}$$

Sum and Product Rules



Sum Rule

$$\begin{aligned} p(X = x_i) &= \frac{c_i}{N} = \frac{1}{N} \sum_{j=1}^L n_{ij} \\ &= \sum_{j=1}^L p(X = x_i, Y = y_j) \end{aligned}$$

Product Rule

$$\begin{aligned} p(X = x_i, Y = y_j) &= \frac{n_{ij}}{N} = \frac{n_{ij}}{c_i} \cdot \frac{c_i}{N} \\ &= p(Y = y_j | X = x_i) p(X = x_i) \end{aligned}$$

Sum and Product Rules

- **Sum Rule** $p(X) = \sum_Y p(X, Y)$
- **Product Rule** $p(X, Y) = p(Y|X)p(X)$

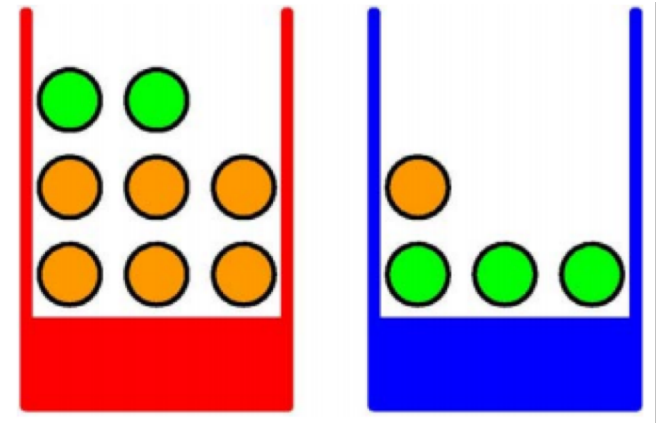
Law of Total Probability

- If an event A can occur in m different ways and if these m different ways are mutually exclusive, then the probability of A occurring is the sum of the probabilities of the sub-events

$$P(X = x_i) = \sum_j P(X = x_i | Y = y_j) P(Y = y_j)$$

Sum and Product Rules

- Back to the fruit baskets
 - $p(B=r)=4/10$ and $p(B=b)=6/10$
 - $p(B=r) + p(B=b) = 1$
- Conditional probabilities
 - $p(F=a \mid B = r) = 1/4$
 - $p(F=o \mid B = r) = 3/4$
 - $p(F=a \mid B = b) = 3/4$
 - $p(F=o \mid B = b) = 1/4$



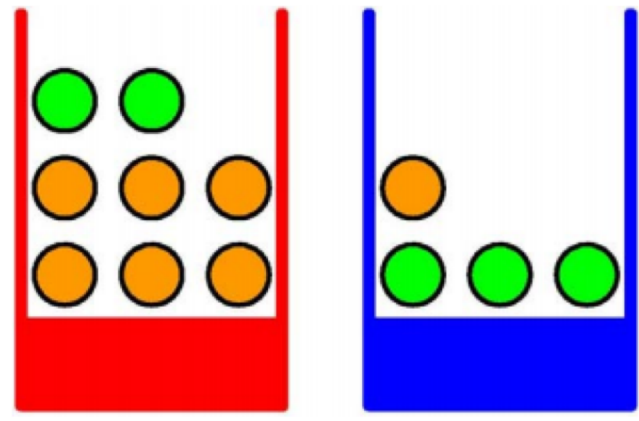
Sum and Product Rules

- Note:

$$p(F=a \mid B=r) + p(F=o \mid B=r) = 1$$

$$\begin{aligned} p(F=a) &= p(F=a \mid B=r) p(B=r) + p(F=a \mid B=b) p(B=b) \\ &= 1/4 * 4/10 + 3/4 * 6/10 = 11/20 \end{aligned}$$

- Sum rule: $p(F=o) = ?$



Outline

- Random variables and statistical independence
- Conditional Probability
- Sum and Product Rules
- **Bayes Rule**

Law of Total Probability

$$P_x(x) = \sum_{y \in Y} P(x, y)$$

$$P(x | y) = \frac{P(x, y)}{P(y)}$$

Bayes Rule

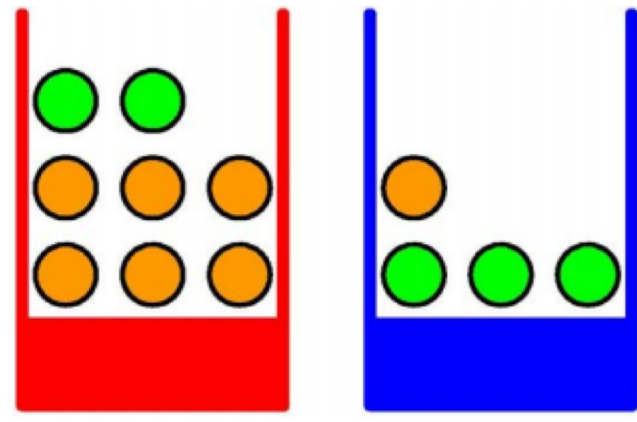
$$P(x | y) = \frac{P(x, y)}{P(y)} = \frac{P(y | x)P(x)}{\sum_{x \in X} P(x, y)}$$
$$\text{posterior} = \frac{\text{likelihood} * \text{prior}}{\text{evidence}}$$

- x is the unknown cause
- y is the observed evidence
- Bayes rule shows how probability of x changes after we have observed y

Bayes Rule on the Fruit Example

- Suppose we have selected an orange. Which box did it come from?

$$p(B = r \mid F = o) = \frac{p(F = o \mid B = r)p(B = r)}{p(F = o)} = \frac{\frac{3}{4} \times \frac{4}{10}}{\frac{9}{20}} = \frac{2}{3}$$



Continuous Random Variables

- Examples: room temperature, time to run 100m, weight of child at birth...
- Cannot talk about probability of that x has a particular value
- Instead, probability that x falls in an interval 
probability density function

$$\Pr[x \in (a, b)] = \int_a^b p(x) dx$$
$$p(x) \geq 0 \text{ and } \int_{-\infty}^{\infty} p(x) dx = 1$$

Bayes Rule for Continuous Case

- Bayes rule
$$p(x | y) = \frac{p(y | x)p(x)}{\int_{-\infty}^{\infty} p(y | x)p(x)dx}$$
$$\text{posterior} = \frac{\text{likelihood} * \text{prior}}{\text{evidence}}$$

Next class

- Topic: Review for Midterm Exam 2.
- Pre-class reading: Chap 4-6.

