UNIVERSITY AT ALBANY
State University of New York

Lecture 9: Sequences and Summations (1)

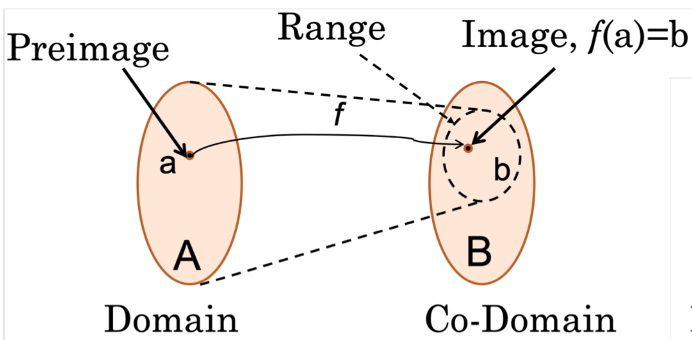
Dr. Chengjiang Long
Computer Vision Researcher at Kitware Inc.
Adjunct Professor at SUNY at Albany.
Email: **clong2@albany.edu**

Office Hours

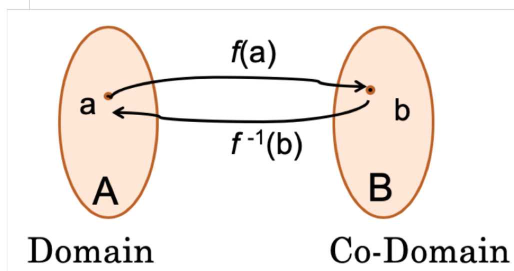
- Instructor’s Office Hour (by appointment)
  - Wed 12:45 PM – 3:45 PM at UAB 412E.
- TA’s Office Hours
  - Siqian, Zhao, Thu 2:50 PM – 5:50 PM at UAB 434.
  - Ronk Osiptan, Tue 1:20 PM - 2:20 PM at UAB 412D.
  - Ronk Osiptan, Thu 1:30 PM - 3:30 PM at UAB 412D.

Recap Previous Lecture

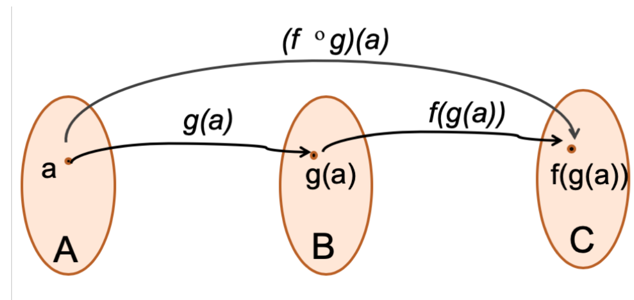
- Properties (One-to-one (injective), onto (surjective), one-to-one correspondence (bijective))
- Inverse functions (examples)
- Operators (Composition, Equality)



A function, f: A → B



A function and its inverse



The composition of two functions

Outline

- Sequences
- Progressions: Special sequences
- Summations

Outline

- **Sequences**
- Progressions: Special sequences
- Summations

Sequence

A sequence is an *ordered* list of elements.

- A sequence is often given as
  - $a_1, a_2, \ldots, a_n, \ldots$
  - a_n is a term in the sequence.

- A sequence is actually a function *f* from a subset of **Z** to a set S
  - Usually from the positive or non-negative integers
  - a_n is the image of n: $f(n) = a_n$

Examples of Sequence

- $a_n = 3n$
  - The terms in the sequence are $a_1, a_2, a_3, \ldots$
  - The sequence $\{a_n\}$ is $\{ 3, 6, 9, 12, \ldots \}$

- $b_n = 2^n$
  - The terms in the sequence are $b_1, b_2, b_3, \ldots$
  - The sequence $\{b_n\}$ is $\{ 2, 4, 8, 16, 32, \ldots \}$

Examples of Sequence

- The difference is in how they grow

- Arithmetic sequences increase by a constant *amount*
  - $a_n = 3n$
  - The sequence is { 3, 6, 9, 12, … }
  - Each number is 3 more than the last
  - Of the form: $f(x) = dx + a$

- Geometric sequences increase by a constant *factor*
  - $b_n = 2^n$
  - The sequence is { 2, 4, 8, 16, 32, … }
  - Each number is twice the previous
  - Of the form: $f(x) = ar^x$

Examples of Sequence

- Sequences can be neither geometric or arithmetic
  - $F_n = F_{n-1} + F_{n-2}$, where the first two terms are 1
    - Alternative, $F(n) = F(n-1) + F(n-2)$
  - Each term is the sum of the previous two terms
  - Sequence: { 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, … }
  - This is the Fibonacci sequence
  - Full formula: $F(n) = \dfrac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{\sqrt{5} \cdot 2^n}$

Examples of Sequence

Not all sequences are arithmetic or geometric sequences.

An example is Fibonacci sequence

- $F_n = F_{n-1} + F_{n-2}$, where the first two terms are 1
  - Alternative, $F(n) = F(n\text{-}1) + F(n\text{-}2)$
- Each term is the sum of the previous two terms
- Sequence: { 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, … }
- This is the Fibonacci sequence

More on Fibonacci Sequence

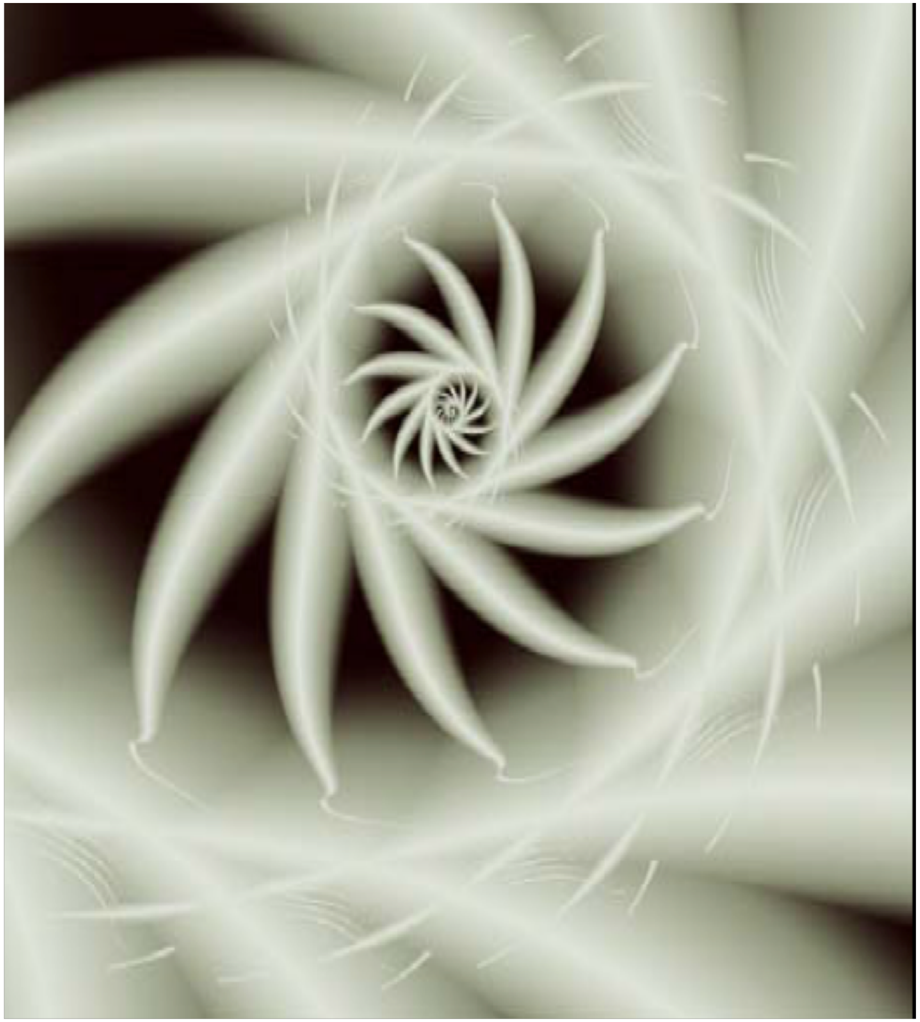
- As the terms increase, the ratio between successive terms approaches 1.618

$$\lim_{n\to\infty}\frac{F(n+1)}{F(n)} = \phi = \frac{\sqrt{5}+1}{2} = 1.618933989$$

- This is called the “golden ratio”
  - Ratio of human leg length to arm length
  - Ratio of successive layers in a conch shell

- Reference: http://en.wikipedia.org/wiki/Golden_ratio

Examples of Golden Ratio



Sequences

- **Definition**: A <u>sequence</u> is a function from a subset of integers to a set S. We use the notation(s):

$$\{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}$$

- Each a_n is called the n^{th} <u>term</u> of the sequence
- We rely on the context to distinguish between a sequence and a set, although they are distinct structures

Outline

- Sequences
- **Progressions: Special sequences**
- Summations

Progressions: Geometric

- **Definition**: A geometric progression is a sequence of the form

$$a, ar, ar^2, ar^3, \ldots, ar^{n-1}$$

  Where:
  - $a \in \mathbb{R}$ is called the initial term
  - $r \in \mathbb{R}$ is called the common ratio
- A geometric progression is a discrete analogue of the exponential function

$$f(x) = ar^x$$

Geometric Progressions: Examples

- A common geometric progression in Computer Science is:

$$\{a_n\}= 1/2^n$$

  with a=1/2 and q=1/2
- Give the initial term and the common ratio of
  - $\{b_n\}$ with $b_n= (-1)^n$
  - $\{c_n\}$ with $c_n= 2(5)^n$
  - $\{d_n\}$ with $d_n= 6(1/3)^n$

Progressions: Arithmetic

- **Definition**: An arithmetic progression is a sequence of the form

$$a, a+d, a+2d, a+3d, \ldots, a+(n-1)d$$

Where:
  - $a \in \mathbb{R}$ is called the initial term
  - $d \in \mathbb{R}$ is called the common difference
- An arithmetic progression is a discrete analogue of the linear function

$$f(x) = dx+a$$

Arithmetic Progressions: Examples

- Give the initial term and the common difference of
  - $\{s_n\}$ with $s_n = -1 + 4n$
  - $\{t_n\}$ with $s_n = 7 – 3n$

Outline

- Sequences
- Progressions: Special sequences
- **Summations**

Summations

- You should be by now familiar with the summation notation:

  $$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_{n-1} + a_n$$

  Here
  - i is the index of the summation
  - m is the lower limit
  - n is the upper limit
- Often times, it is useful to change the lower/upper limits, which can be done in a straightforward manner (although we must be very careful):

  $$\sum_{i=1}^{n} a_i = \sum_{i=0}^{n-1} a_{i+1}$$

Infinite Series: Geometric Series

- In fact, we can generalize that fact as follows
- **Lemma**: A geometric series converges <u>if and only if</u> the absolute value of the common ratio is less than 1

$$S = \begin{cases} a(1-r^n)/(1-r) & \text{if } r \neq 1 \\ na & \text{if } r = 1 \end{cases}$$

Summation of Geometric Sequence

With 5 terms of the general geometric sequence, we have

$$S_5 = a + ar + ar^2 + ar^3 + ar^4$$

TRICK Multiply by *r:*

$$rS_5 = ar + ar^2 + ar^3 + ar^4 + ar^5$$

Subtracting the expressions gives

$$\begin{aligned} S_5 - rS_5 = \quad & a + ar + ar^2 + ar^3 + ar^4 \\ - \quad & \underline{ar + ar^2 + ar^3 + ar^4 + ar^5} \end{aligned}$$

Move the lower row 1 place to the right and subtract

Summation of Geometric Sequence

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$$S_5 - rS_5 = a \qquad\qquad - ar^5$$

Summation of Geometric Sequence

So, $$S_5 - rS_5 = a - ar^5$$

Take out the common factors

$$S_5(1-r) = a(1-r^5)$$

and divide by (1 – *r*)

$$\Rightarrow \qquad S_5 = \frac{a(1-r^5)}{1-r}$$

Similarly, for *n* terms we get

$$S_n = \frac{a(1-r^n)}{1-r}$$

Summation of Geometric Sequence

The formula

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

gives a negative denominator if $r > 1$

Instead, we can use

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Example

Find the sum of the first 20 terms of the geometric series, leaving your answer in index form $2 - 6 + 18 - 54 + \ . \ . \ .$

Solution: $a = 2, \quad r = \dfrac{\overset{-3}{\cancel{-6}}}{\underset{1}{\cancel{2}}} = -3$

$$S_n = \frac{a(1 - r^n)}{1 - r} \Rightarrow S_{20} = \frac{2\left(1 - (-3)^{20}\right)}{1 - (-3)}$$

We'll simplify this answer without using a calculator

Example

$$\Rightarrow \quad S_{20} = \frac{2\left(1-(-3)^{20}\right)}{1-(-3)}$$

There are 20 minus signs here and 1 more outside the bracket!

$$= \frac{\cancel{2}^{\,1}\left(1-3^{20}\right)}{\cancel{4}_{\,2}}$$

$$= \frac{1-3^{20}}{2}$$

Sum of arithmetic series

Given n numbers, $a_1, a_2, \ldots, a_n$ with common difference d, i.e. $a_{i+1} - a_i = d$.

What is a simple closed form expression of the sum?

$$S_n = \sum_{i=1}^{n} a_i$$

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots\ldots + (a_1 + (n-2)d) + (a_1 + (n-1)d)$$

$$S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_n - (n-2)d) + (a_n - (n-1)d)$$

Adding the equations together gives:

$$2S_n = n(a_1 + a_n)$$

Rearranging and remembering that $a_n = a_1 + (n-1)d$, we get:

$$\boxed{S_n = \frac{n(a_1 + a_n)}{2} = \frac{n[2a_1 + (n-1)d]}{2}}$$

Sum of arithmetic series

What is the sum of this series?

$$\sum_{p=1}^{50}(73-2p)$$

$$=71+69+67+\ldots+(-25)+(-27)$$

- Write the first three terms and the last two terms of the following arithmetic series.

Sum of arithmetic series

What is the sum of these terms?

$$71+69+67+\ldots+(-25)+(-27)$$

Written 1st to last.

$$\underline{(-27)+(-25)+\ldots+67+69+71}$$

Written last to 1st.

$$44+44+44+\ldots+44+44+44$$

Add Down

50 Terms

$$=\frac{50(71+(-27))}{2}$$

$$=1100$$

71 + (-27) Each sum is the same.

Sum of arithmetic series

In general

$$a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + (a_1 + (n-1)d)$$

$$(a_1 + (n-1)d) + \ldots + (a_1 + 2d) + (a_1 + d) + a_1$$

$$(a_1 + [a_1 + (n-1)d]) + (a_1 + [a_1 + (n-1)d]) + \ldots + (a_1 + [a_1 + (n-1)d])$$

$$S = \frac{n(a_1 + a_n)}{2} \qquad \begin{cases} S = \text{Sum} \\ n = \text{Number of Terms} \\ a_1 = \text{First Term} \\ a_n = \text{Last Term} \end{cases}$$

Example

$$151+147+143+139+\ldots+(-5)$$

What term is -5?

$$S=\frac{n(a_1+a_n)}{2}$$

$$a_n=a_1+(n-1)d$$

$$-5=151+(n-1)(-4)$$

$$n=40$$

$$\boxed{\begin{aligned} n &= 40 \\ a_1 &= 151 \\ a_{40} &= -5 \end{aligned}}$$

$$\boxed{\begin{aligned} S &= \frac{40(151+-5)}{2} \\ S &= 2920 \end{aligned}}$$

Sum of arithmetic series

Alternate formula for the sum of an Arithmetic Series.

Substitute $a_n = a_1 + (n-1)d$

$$S = \frac{n(a_1 + a_n)}{2}$$

$$S = \frac{n(a_1 + a_1 + (n-1)d)}{2}$$

$$S = \frac{n(2a_1 + (n-1)d)}{2}$$

$$\begin{cases} n = \text{\# of Terms} \\ a_1 = \text{1st Term} \\ d = \text{Difference} \end{cases}$$

Solve this

$$\sum_{j=0}^{36}(2.25+0.75j) = 2.25+3+3.73+4.5+\ldots$$

$$S = \frac{n\left(2a_1+(n-1)d\right)}{2}$$

It is not convenient to find the last term.

$$n = 37$$
$$a_1 = 2.25$$
$$d = 0.75$$

$$S = \frac{37\left(2(2.25)+(37-1)(0.75)\right)}{2}$$

$$S = 582.75$$

Next class

- Topic: Sequences and Summation (2)
- Pre-class reading: Chap 2.4