UNIVERSITY AT ALBANY
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Lecture 35: Graph Theory and Trees

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Outline

- Planar Graphs
- Graph Coloring
- Introduction to Trees
- Tree Traversal
- Spanning Trees

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- **Planar Graphs**
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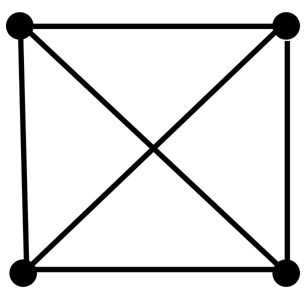
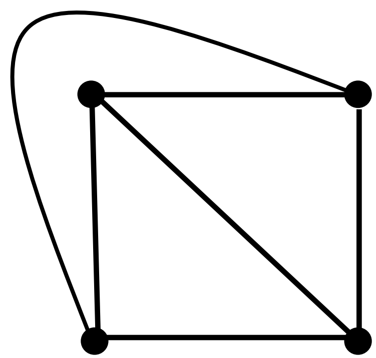
Planar Graphs

- A graph is called *planar* if it can be drawn in the plane without any edge crossing. Such a drawing is called a *planar representation* of the graph.

- Is K_4 planar?



K_4

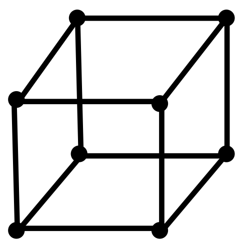


K_4 drawn with no crossings

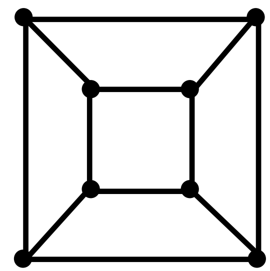
$\therefore$ K_4 is planar

Example

- Is Q_3 planar?



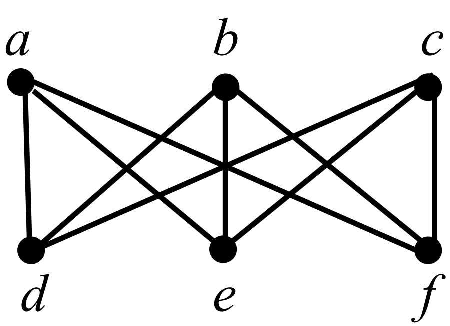
Q_3



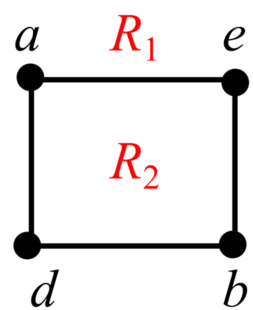
Q_3 drawn with no crossings

$\therefore$ Q_3 is planar

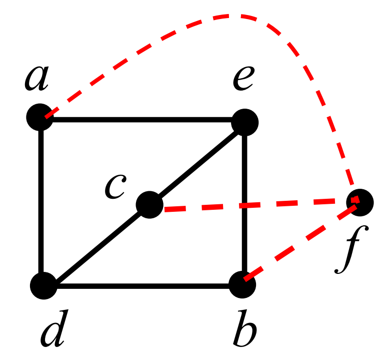
- Show that $K_{3,3}$ is nonplanar.



Solution:



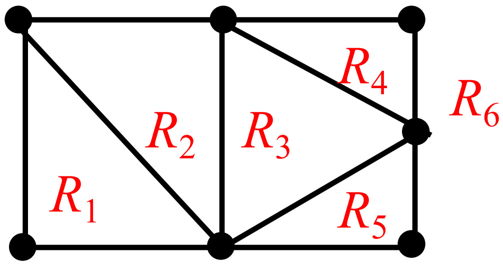
In any drawing, aebd is cycle, and will cut the plane into two regions



Regardless of which region c, could no longer put the f in that side staggered

Euler's Formula

- A planar representation of a graph splits the plane into regions, including an unbounded region.
- **Example :** How many regions are there in the following graph?



Solution: 6

Euler's Formula

Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

Example

- Suppose that a connected planar graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?
- **Solution:**

$$v = 20,\ 2e = 3 \times 20 = 60,\ e = 30$$

$$r = e - v + 2 = 30 - 20 + 2 = 12$$

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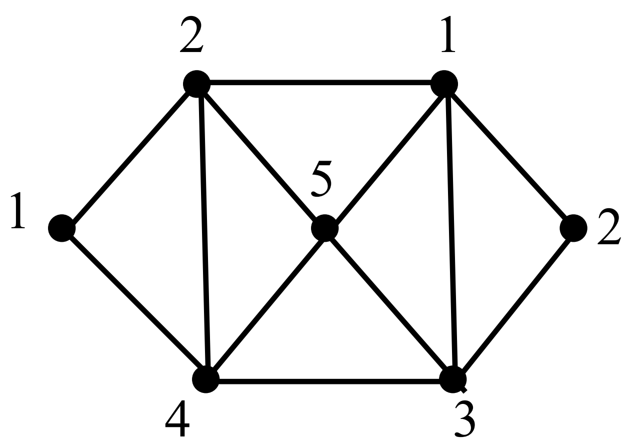
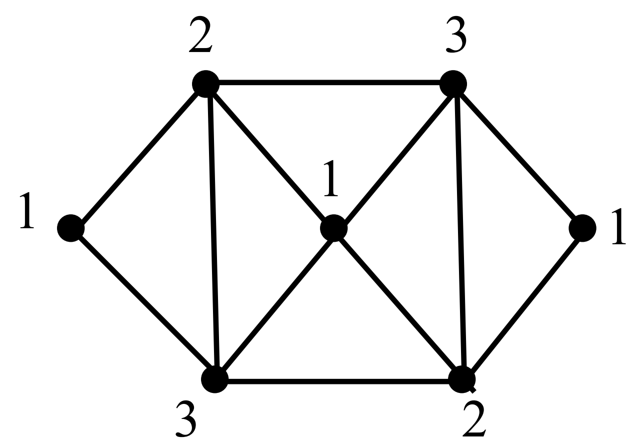
Graph Coloring

- A *coloring* of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Example:



5-coloring



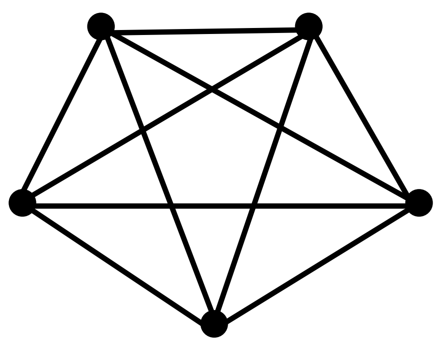
3-coloring

Less the number of colors, the better

Graph Coloring

- The *chromatic number* of a graph is the least number of colors needed for a coloring of this graph. (denoted by $\chi(G)$)

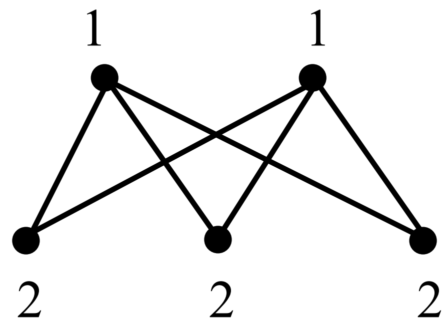
Example: $\chi(K_5)=5$



Note: $\chi(K_n)=n$

Example

$\chi(K_{2,3}) = 2.$

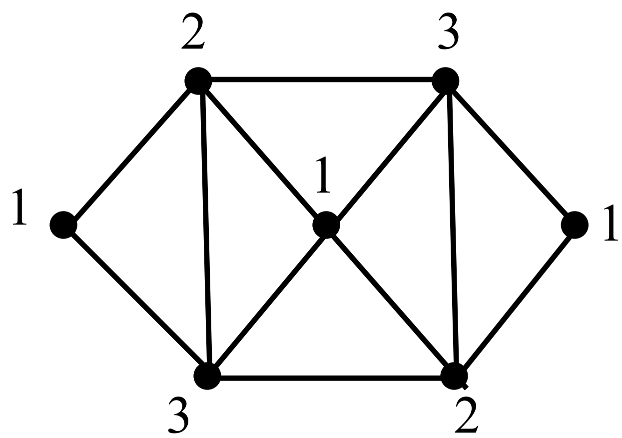


Note: $\chi(K_{m,n}) = 2$

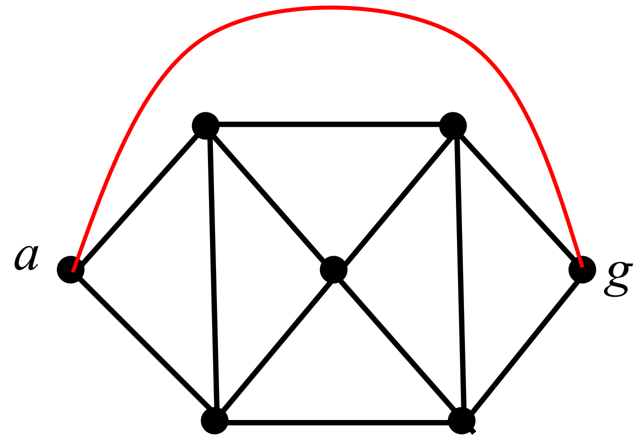
Note: If G is a bipartite graph, $\chi(G) = 2.$

Example

- What are the chromatic numbers of the graphs G and H?



G



H

Solution: G has a 3-cycle $\Rightarrow \chi(G) \geq 3$

G has a 3-coloring $\Rightarrow \chi(G) \leq 3$

$\Rightarrow \chi(G) = 3$

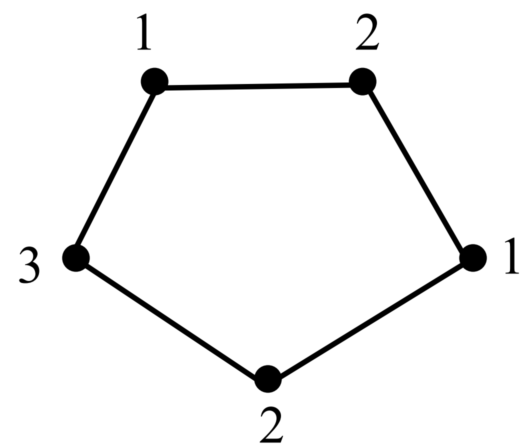
Solution: any 3-coloring for $H - \{(a,g)\}$ gives the same color to a and g $\Rightarrow \chi(H) > 3$

4-coloring exists $\Rightarrow \chi(H) = 4$

Example

$\chi(C_n) = 2$ if n is even, $\chi(C_n) = 3$ if n is odd.

C_n is bipartite
when n is even.



Theorem 1. (The Four Color Theorem)
The chromatic number of a planar graph is no greater than four.

Corollary
Any graph with chromatic number >4 is nonplanar.

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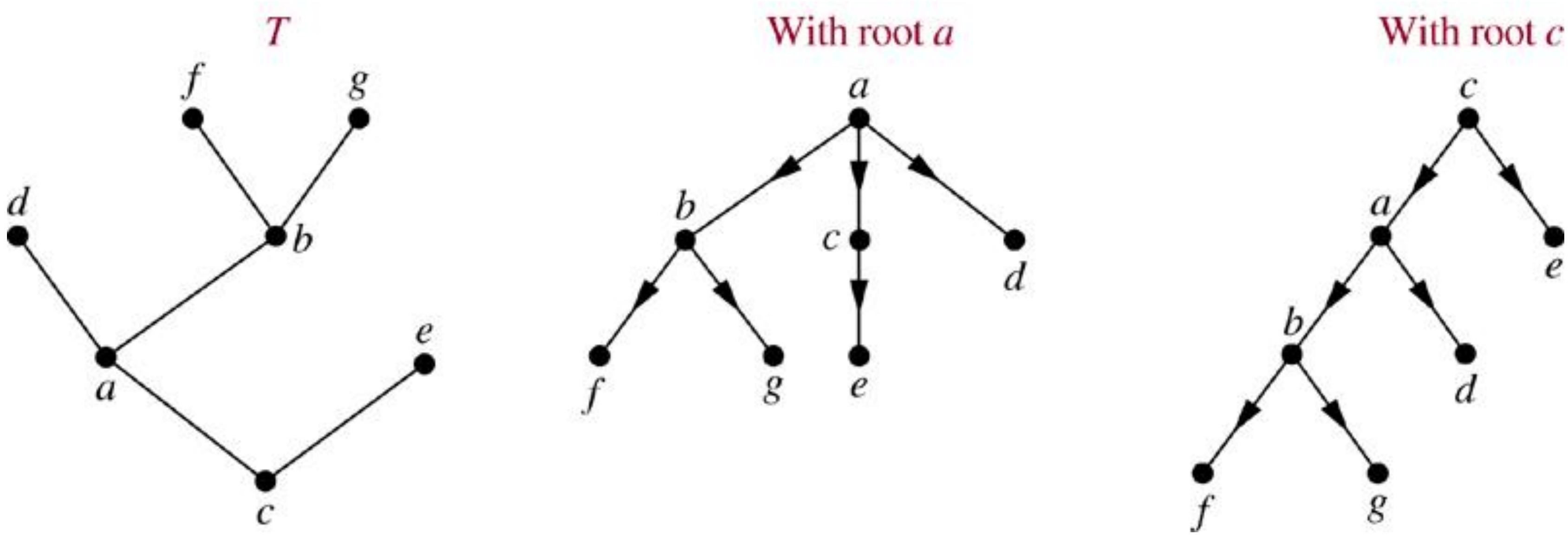
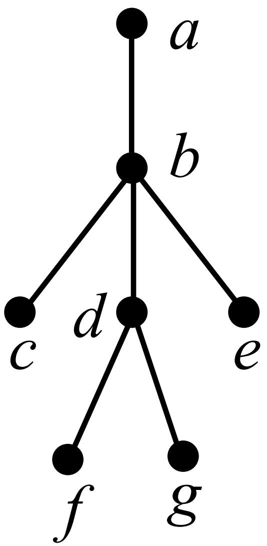
Introduction to Trees

- A tree is a connected undirected graph with no simple circuits.
- A rooted tree is a tree in which one vertex has been designed as the root and every edge is directed away from the root.



Introduction to Trees



a is the parent of b, b is the child of a,

c, d, e are siblings,

a, b, d are ancestors of f

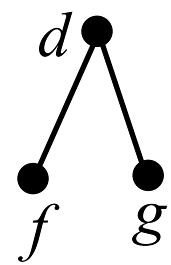
c, d, e, f, g are descendants of b

c, e, f, g are leaves of the tree (deg=1)

a, b, d are internal vertices of the tree

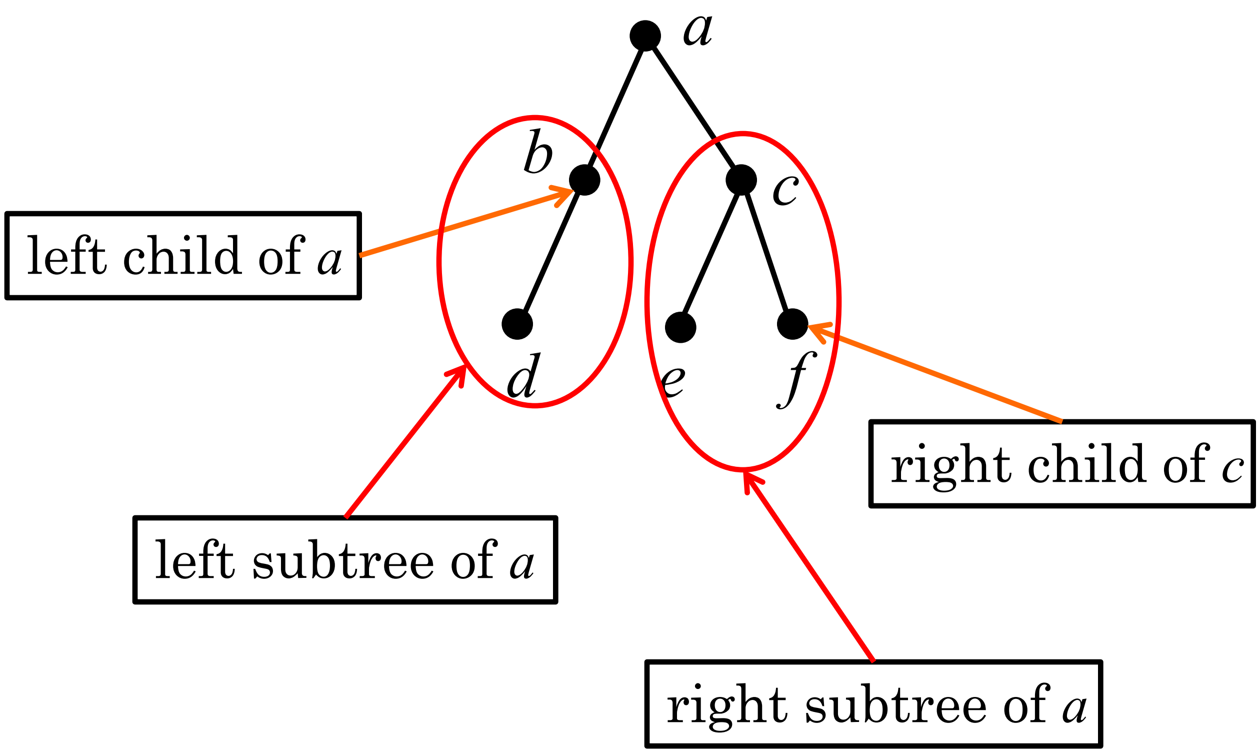
(at least one child)

subtree with d as its root:



Vertices that have children are called **internal vertices.**

Binary Tree

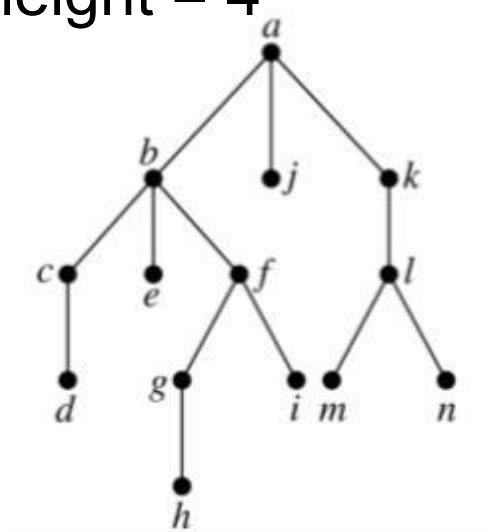


a
b
c
d
e
f
left child of a
right child of c
left subtree of a
right subtree of a

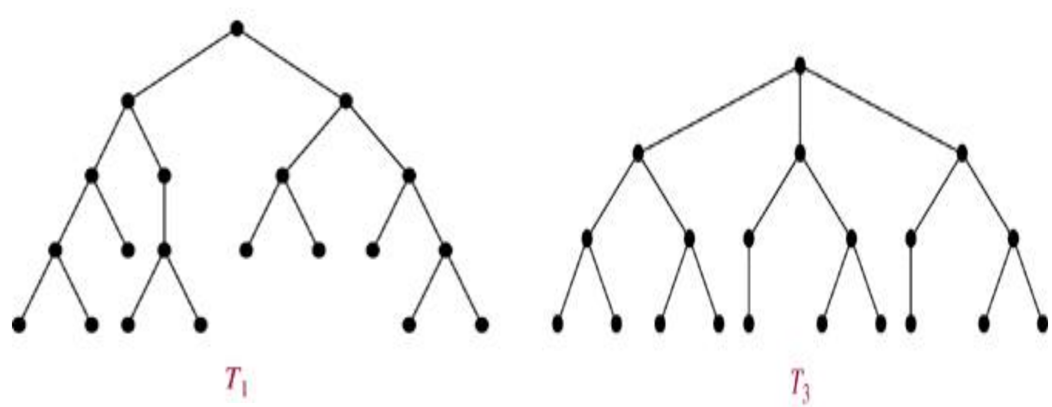
Levels of Trees

- The level of a vertex v in a rooted tree is the length of the unique path from the root to this vertex.
  The level of the root is defined to be zero.
  The height of a rooted tree is the maximum of the levels of vertices.

height = 4



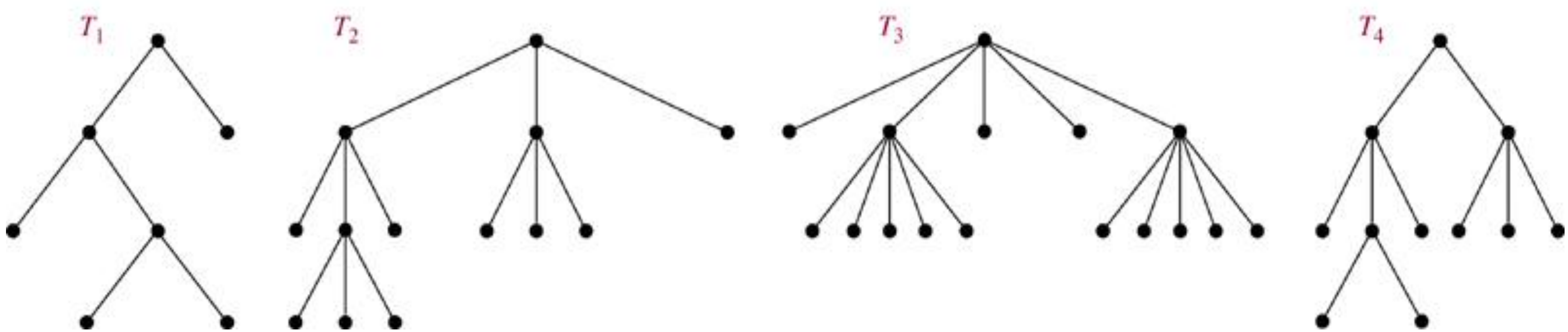
level
0
1
2
3
4



Remark: A rooted m-ary tree of height h is balanced if all leaves are at levels h or $h-1$.

m-ary Tree

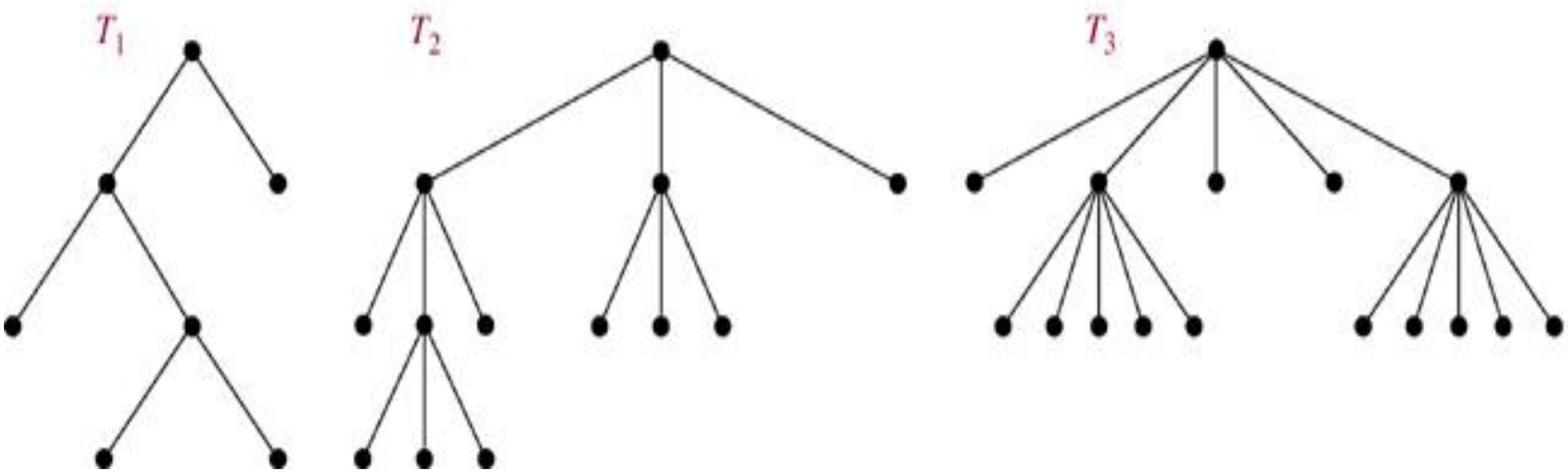
- A rooted tree is called an m-ary tree if every internal vetex has no more than m children.
- The tree is called a full m-ary tree if every internal vertex has exactly m children. An m-ary tree with m=2 is called a binary tree.



full binary tree | full 3-ary tree | full 5-ary tree | not full 3-ary tree

Properties of Trees

- A tree with n vertices has n-1 edges.
- A full m-ary tree with i internal vertices contains $n = mi + 1$ vertices.
- A full m-ary tree with n vertices contains $(n-1)/m$ internal vertices, and hence $n - (n-1)/m = ((m-1)n+1)/m$ leaves.



Complete m-ary Tree

- A complete m-ary tree is a full m-ary tree, where every leaf is at the same level.

Example: How many vertices and how many leaves does a complete m-ary tree of height h have?

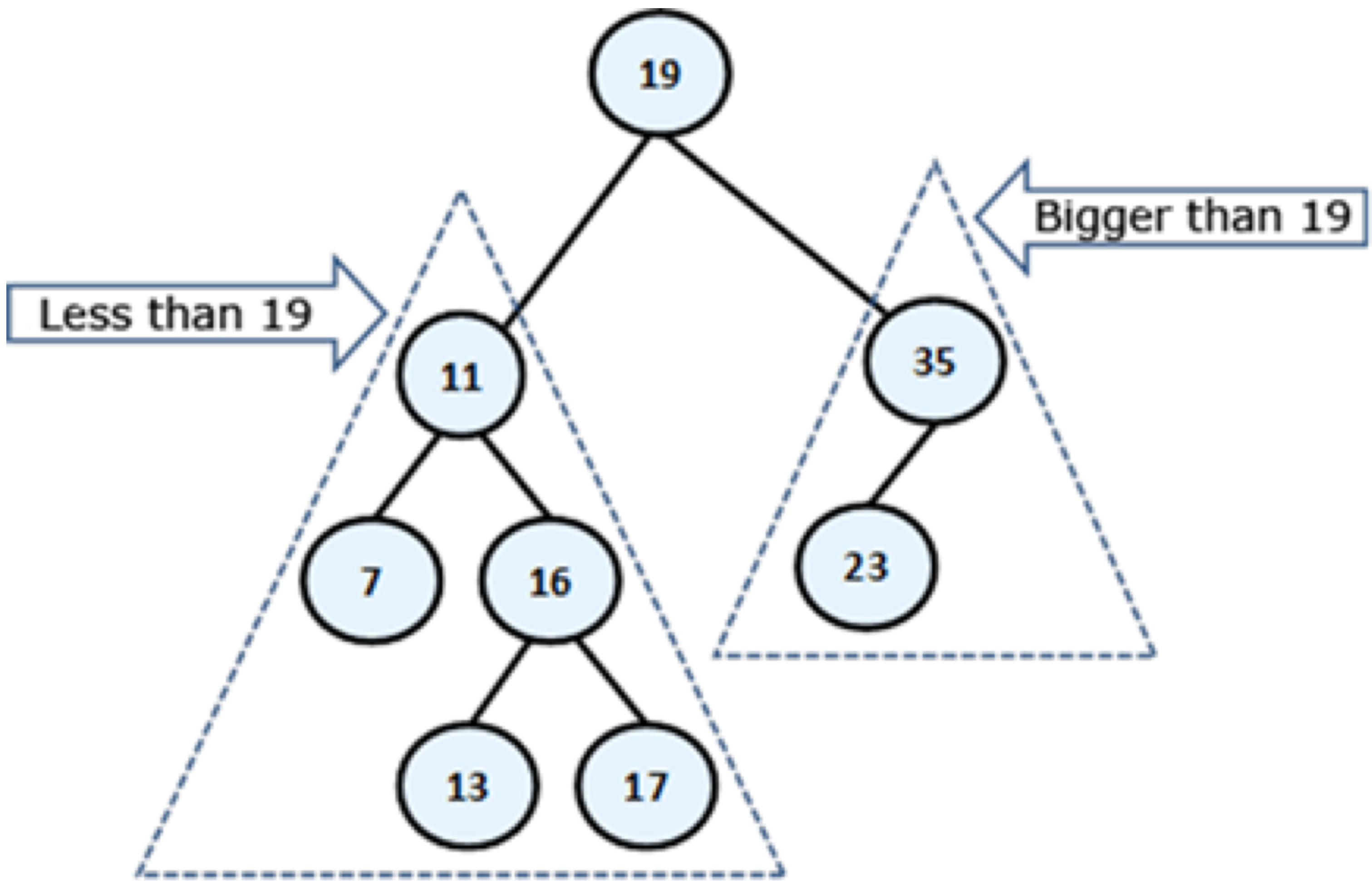
Solution:

# of vertices = $1+m+m^2+\ldots+m^h = (m^{h+1}-1)/(m-1)$

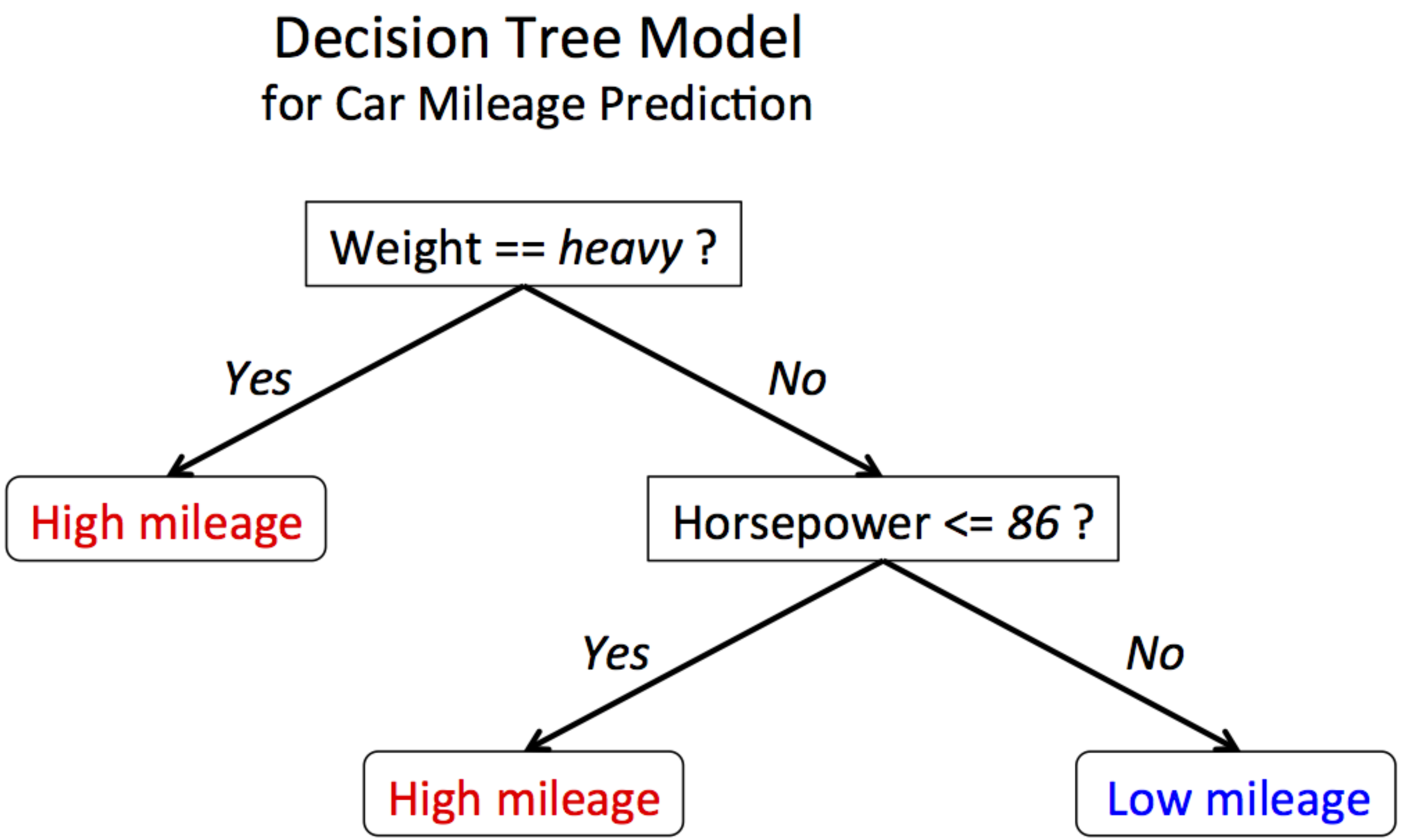
# of leaves = m^h

Remark: There are at most m^h leaves in an m-ary tree of height h.

Applications: Binary Search Trees



Applications: Decision Trees



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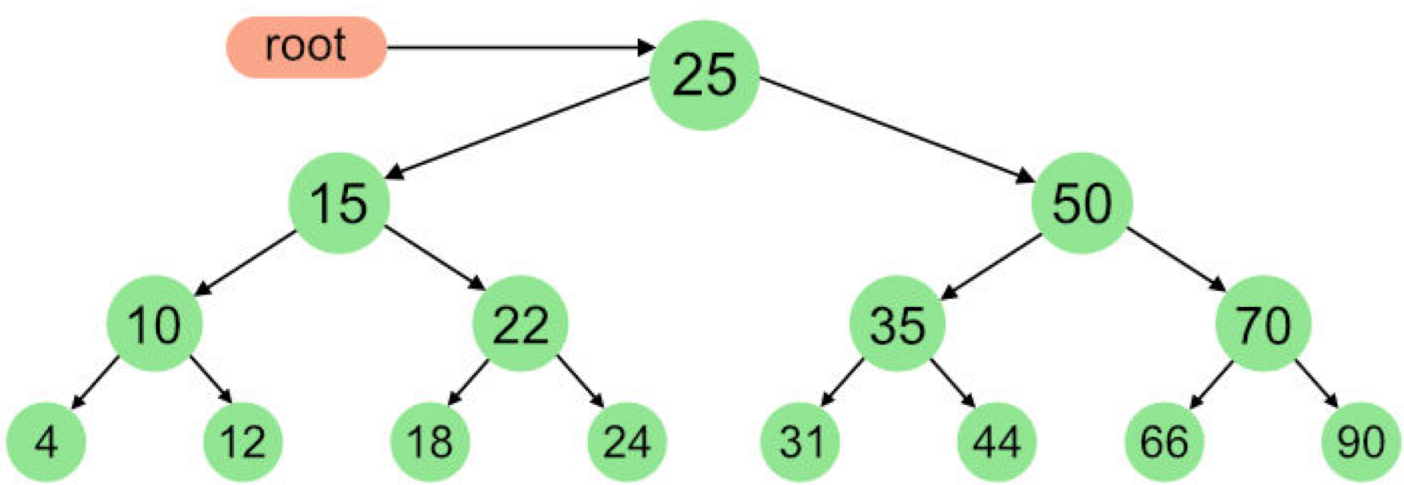
Tree Traversal

- We need procedures for visiting each vertex of an ordered rooted tree to access data.



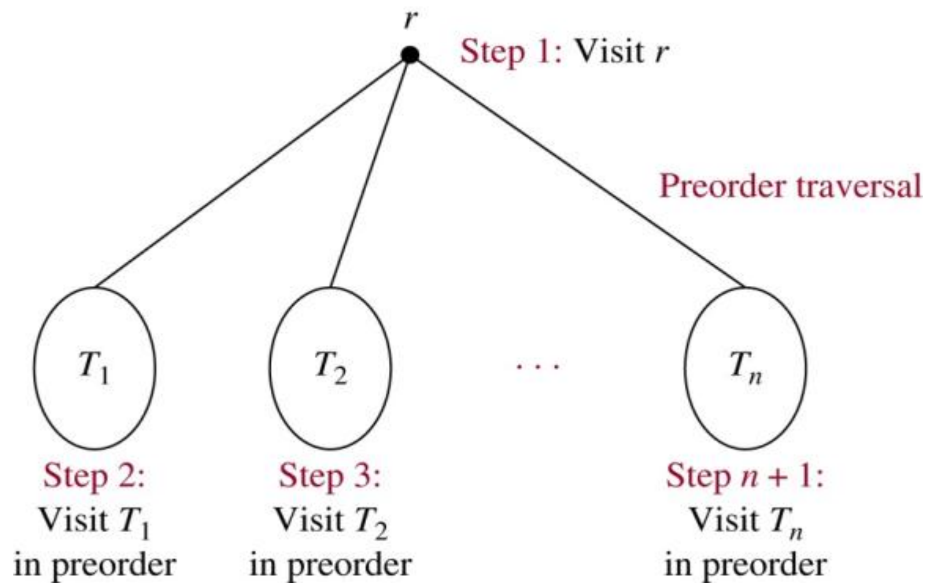
A Pre-order traversal visits nodes in the following order:
25, 15, 10, 4, 12, 22, 18, 24, 50, 35, 31, 44, 70, 66, 90

InOrder(root) visits nodes in the following order:
4, 10, 12, 15, 18, 22, 24, 25, 31, 35, 44, 50, 66, 70, 90

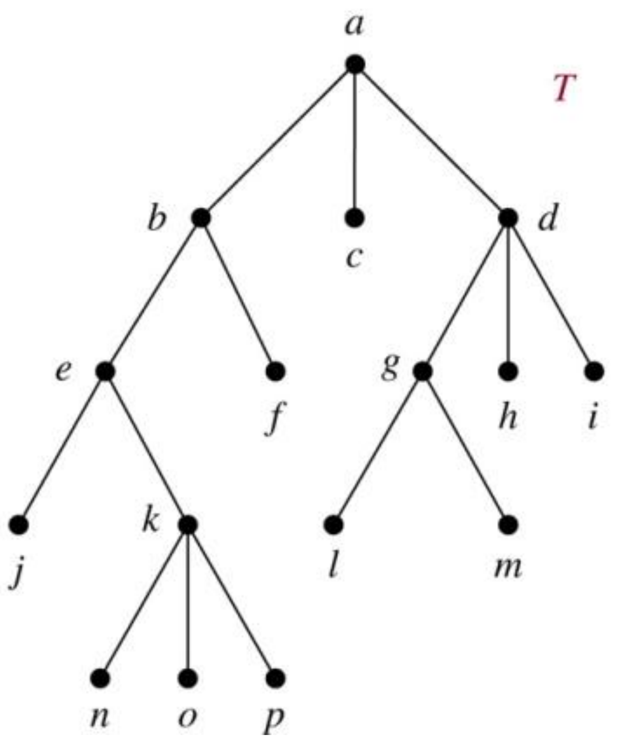
A Post-order traversal visits nodes in the following order:
4, 12, 10, 18, 24, 22, 15, 31, 44, 35, 66, 90, 70, 50, 25

Preorder traversal (Preorder)

Procedure *preorder*(T: ordered rooted tree)
r := root of T
list r
for each child c of r from left to right
begin
    $T(c)$:= subtree with c as its root
    preorder($T(c)$)
end



Example


a
T
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f
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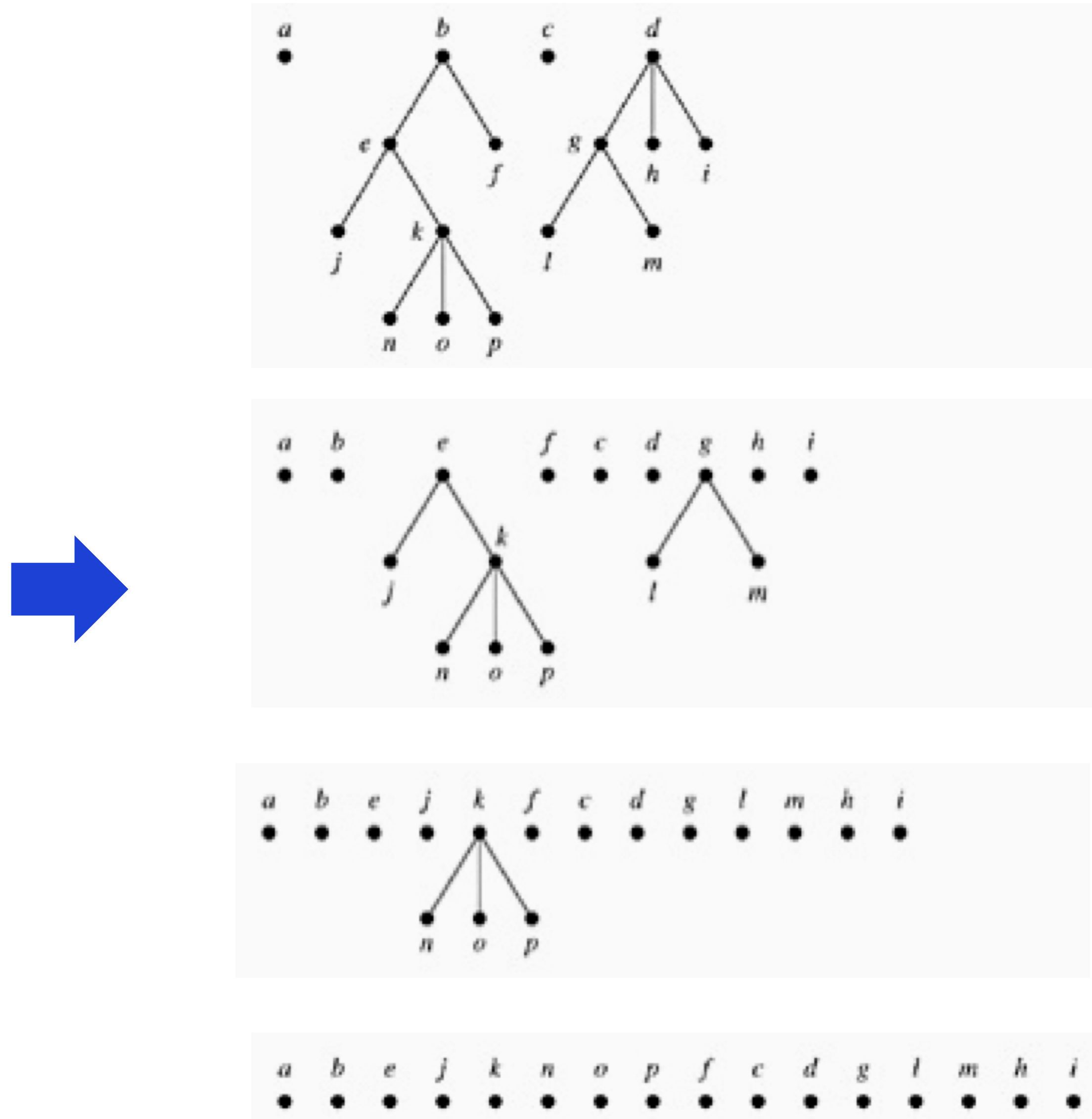
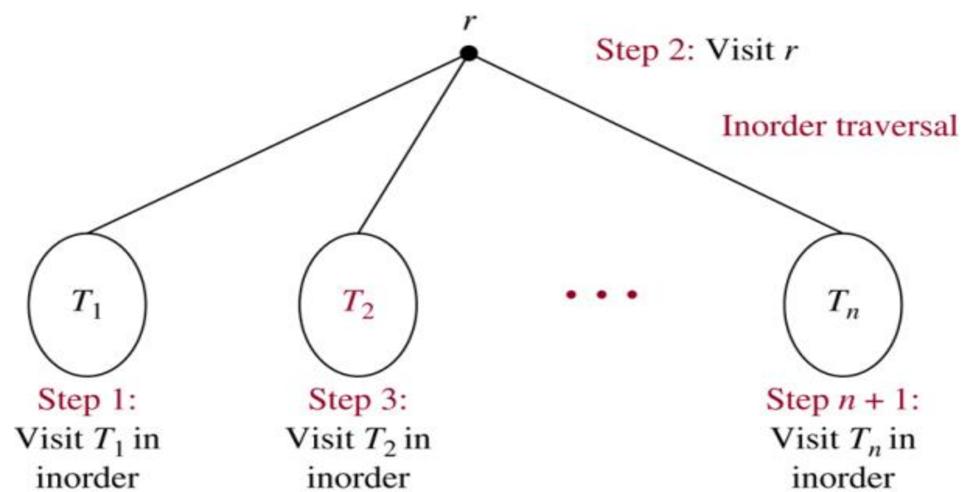
h
i
j
k
l
m
n
o
p

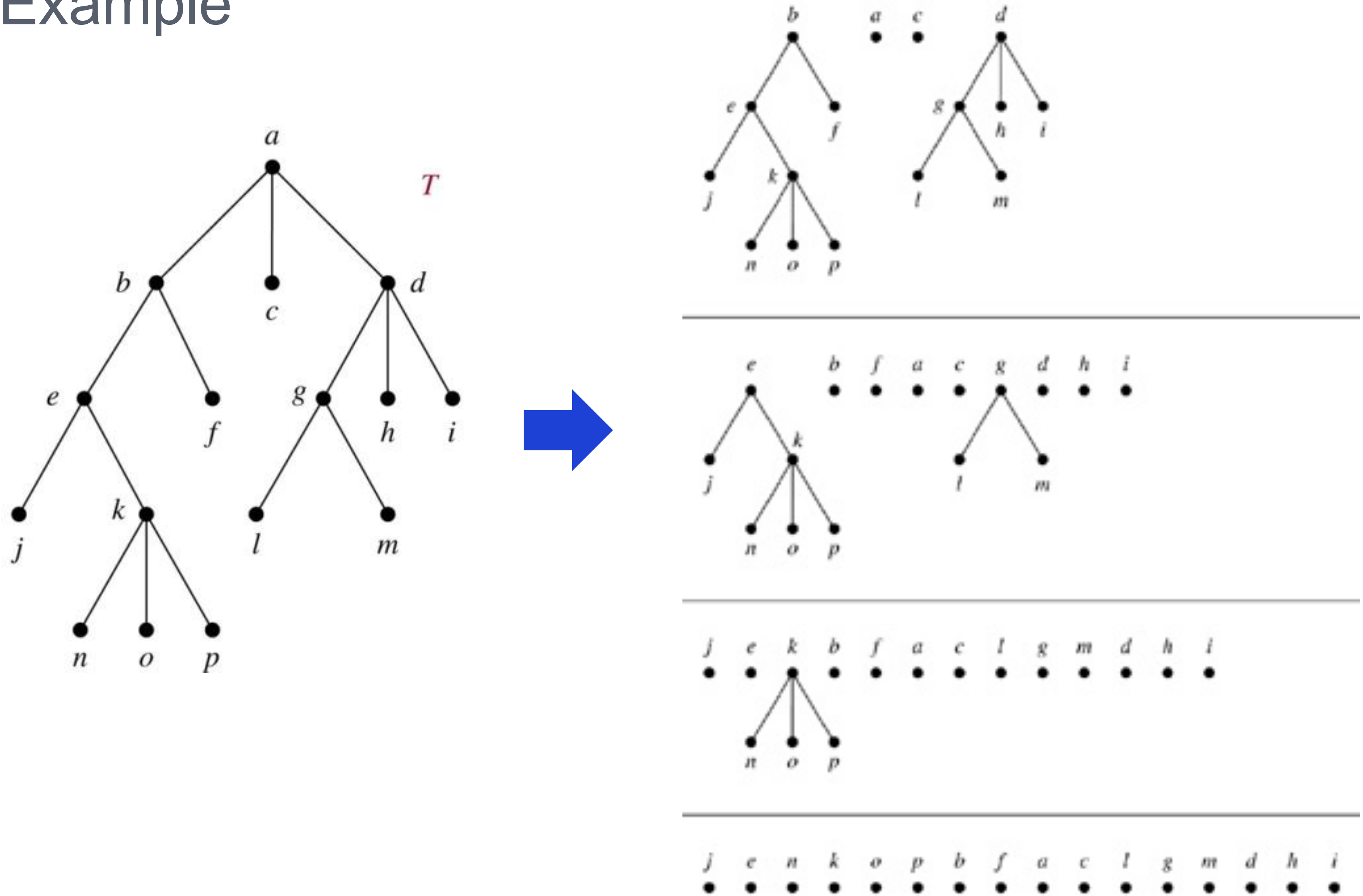

a b c d
e f g h i
j k l m
n o p
a b e f c d g h i
k
j l m
n o p
a b e j k f c d g l m h i
n o p
a b e j k n o p f c d g l m h i

Inorder traversal (Ineorder)

Procedure *inorder*(T: ordered rooted tree)
r := root of T
If r is a leaf **then** list r
else
begin
  l := first child of r from left to right
  $T(l)$:= subtree with l as its root
  inorder($T(l)$)
  list r
  for each child c of r except for l from left to right
    $T(c)$:= subtree with c as its root
    inorder($T(c)$)
end

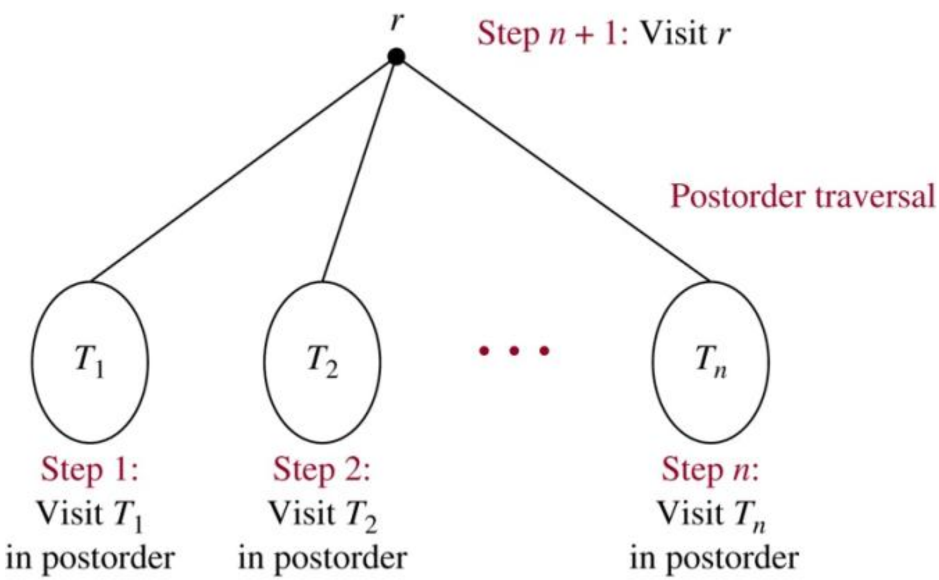


Example

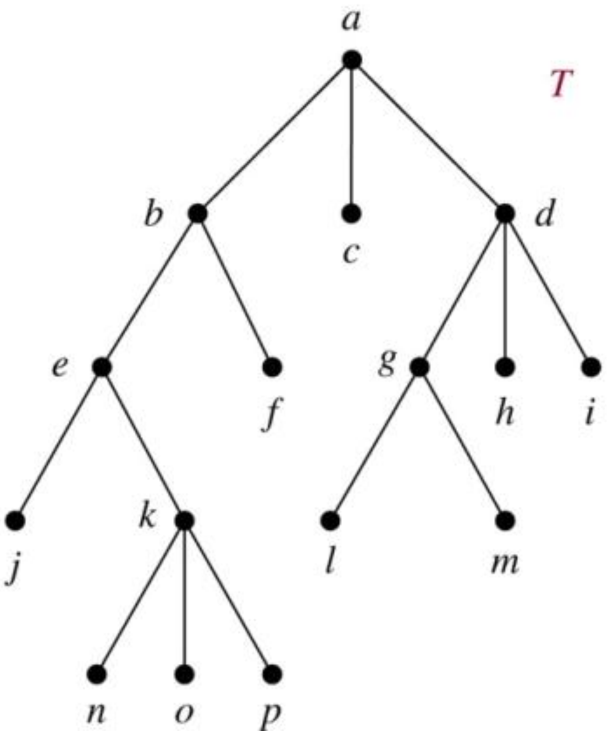


Postorder traversal (Postorder)

Procedure *postorder*(T: ordered rooted tree)
r := root of T
for each child c of r from left to right
begin
    $T(c)$:= subtree with c as its root
    postorder($T(c)$)
end
list r



Example



a
T
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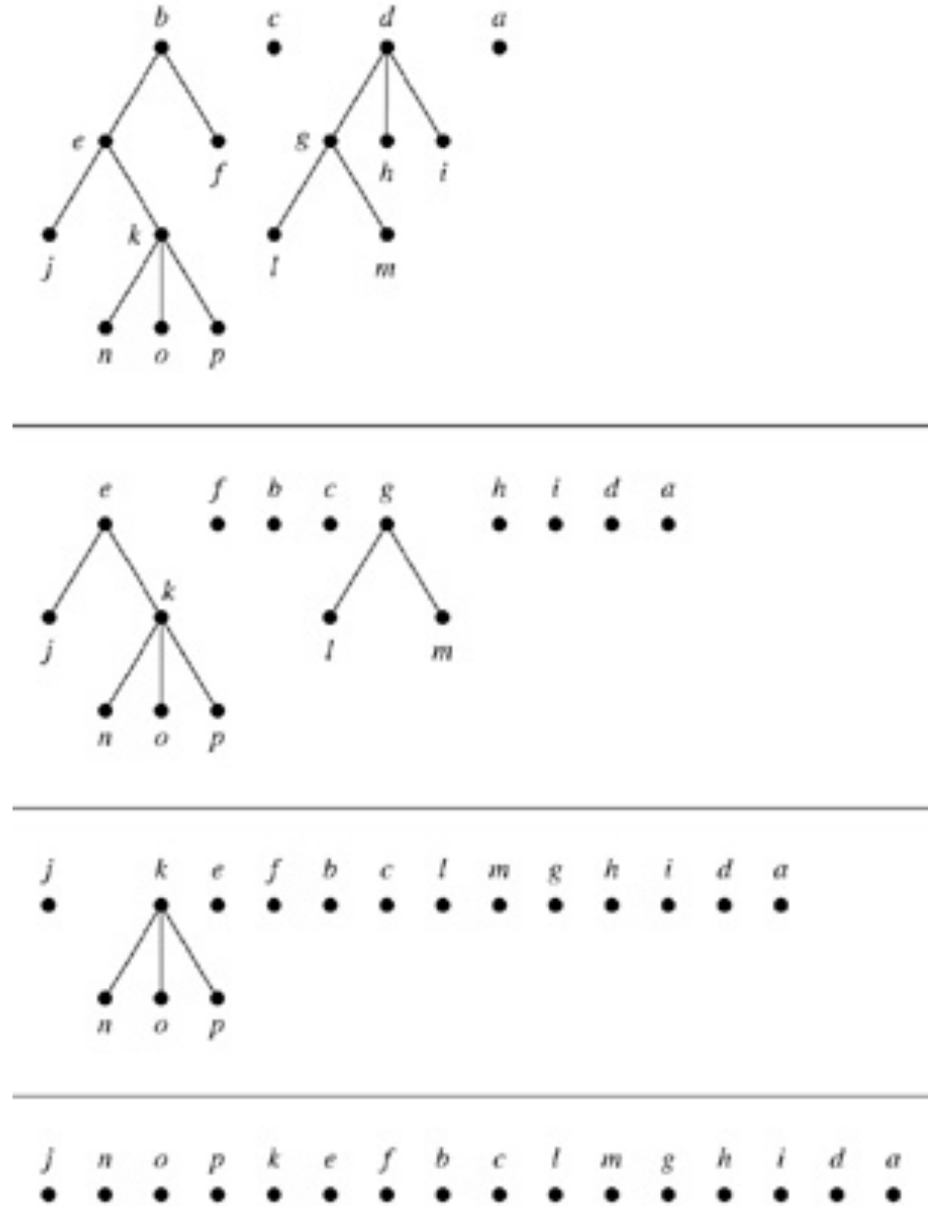
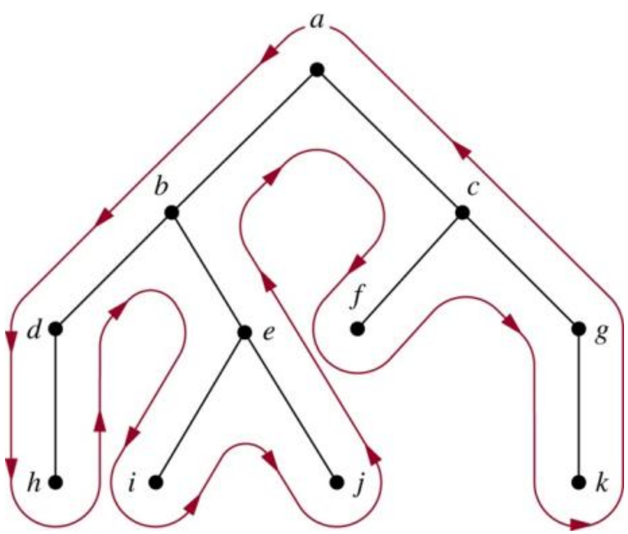


b c d a
e f g h i
j k l m
n o p
e f b c g h i d a
j k l m
n o p
j k e f b c l m g h i d a
n o p
j n o p k e f b c l m g h i d a

Tree Traversal Representation

- Easy representation: draw a red line around the ordered rooted tree starting at the root, moving along the edges.



Preorder: $a, b, d, h, e, i, j, c, f, g, k$

Inorder: $h, d, b, i, e, j, a, f, c, k, g$

Postorder: $h, d, i, j, e, b, f, k, g, c, a$

- Preorder: listing each vertex the first time this line passes it.
- Inorder: listing a leaf the first time the line passes it and listing each internal vertex the second time the line passes it.
- Postorder: listing a vertex the last time it is passed on the way back up to its parent.

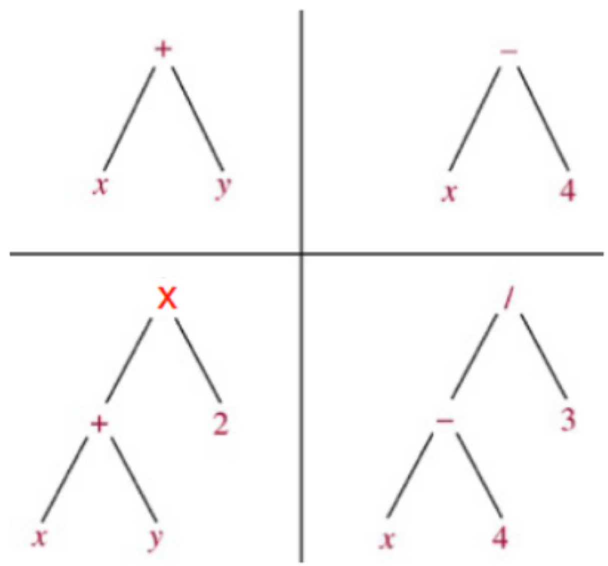
Example

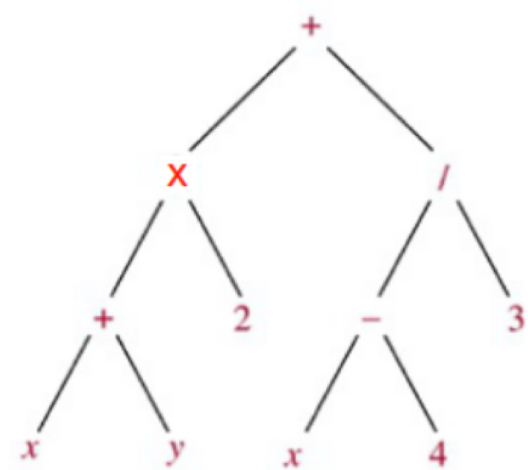
- We can represent complicated expressions, such as compound propositions, combinations of sets, and arithmetic expressions using ordered rooted trees.

Example Find the ordered rooted tree for $((x+y) \times 2)+((x-4)/3)$.

Solution:

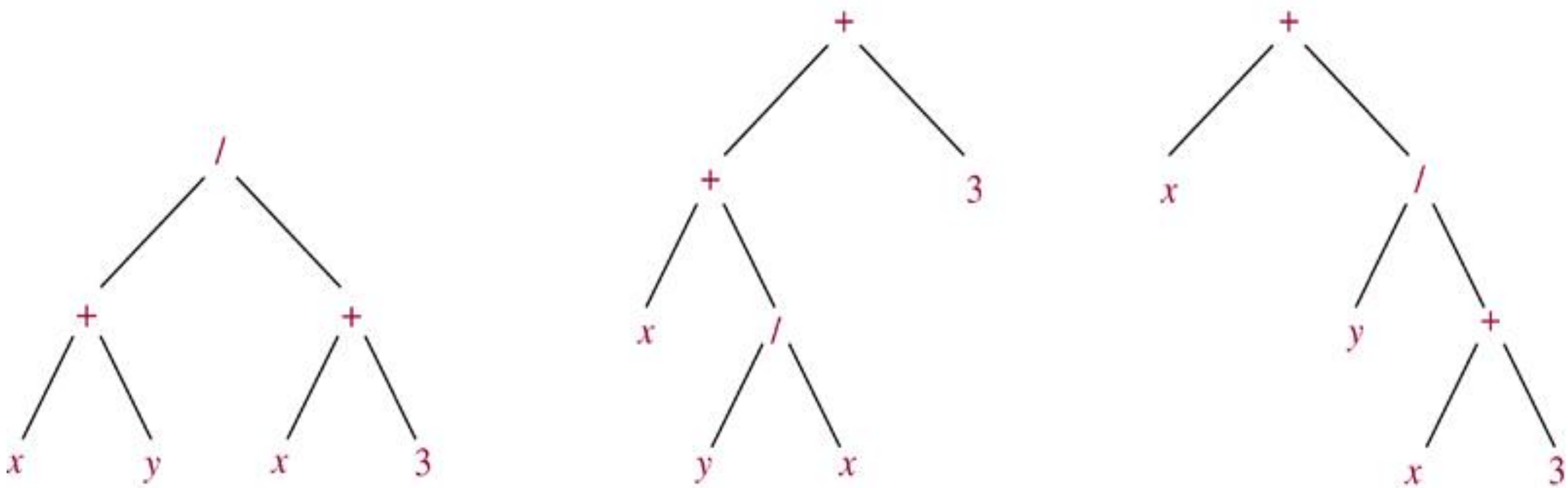
leaf:
  variable
internal vertex:
  operation on its left and right subtrees





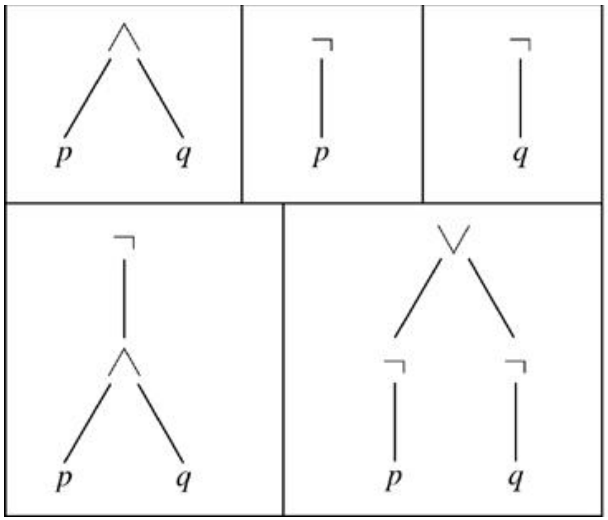
Example

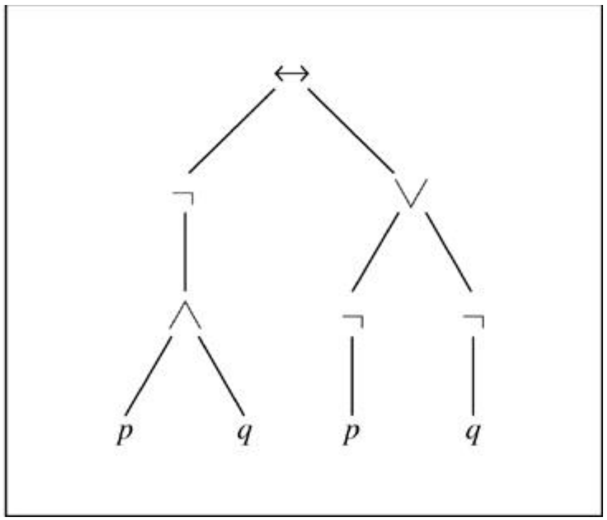
- The following binary trees represent the expressions: (x+y)/(x+3), (x+(y/x))+3, x+(y/(x+3)).
All their inorder traversals lead to x+y/x+3 ⇒ ambiguous ⇒ need parentheses



Example

- Find the ordered rooted tree representing the compound proposition $(\neg(p \wedge q)) \leftrightarrow (\neg p \vee \neg q)$. Then use this rooted tree to find the prefix, postfix, and infix forms of this expression.





prefix: $\leftrightarrow \neg \wedge p\ q \vee \neg p \neg q$

infix: $(\neg(p \wedge q)) \leftrightarrow ((\neg p) \vee (\neg q))$

postfix: $p\ q \wedge \neg p \neg q \neg \vee \leftrightarrow$

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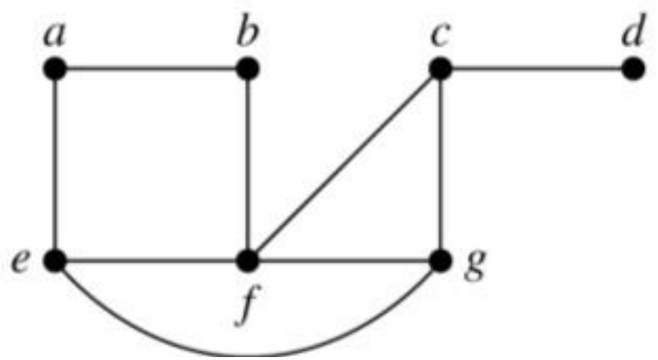
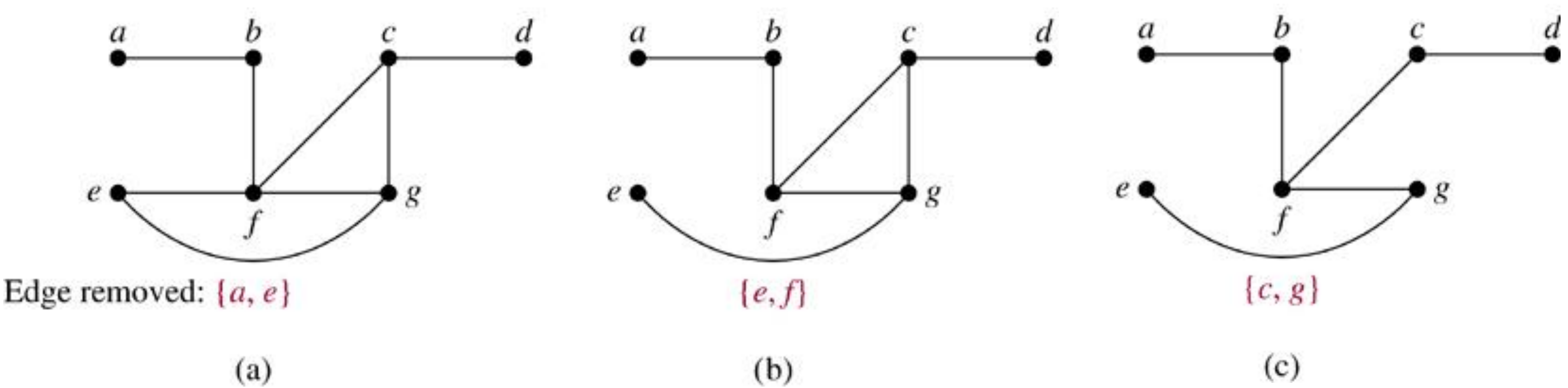
Spanning Trees

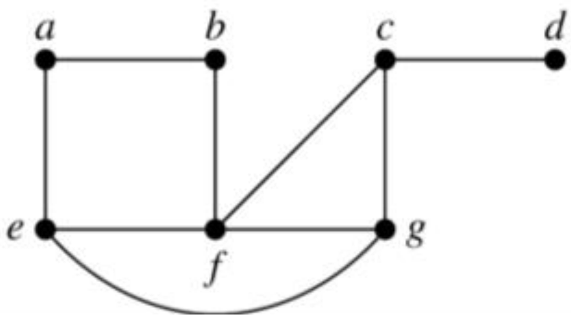
- Let G be a simple graph. A spanning tree of G is a subgraph of G that is a tree containing every vertex of G.



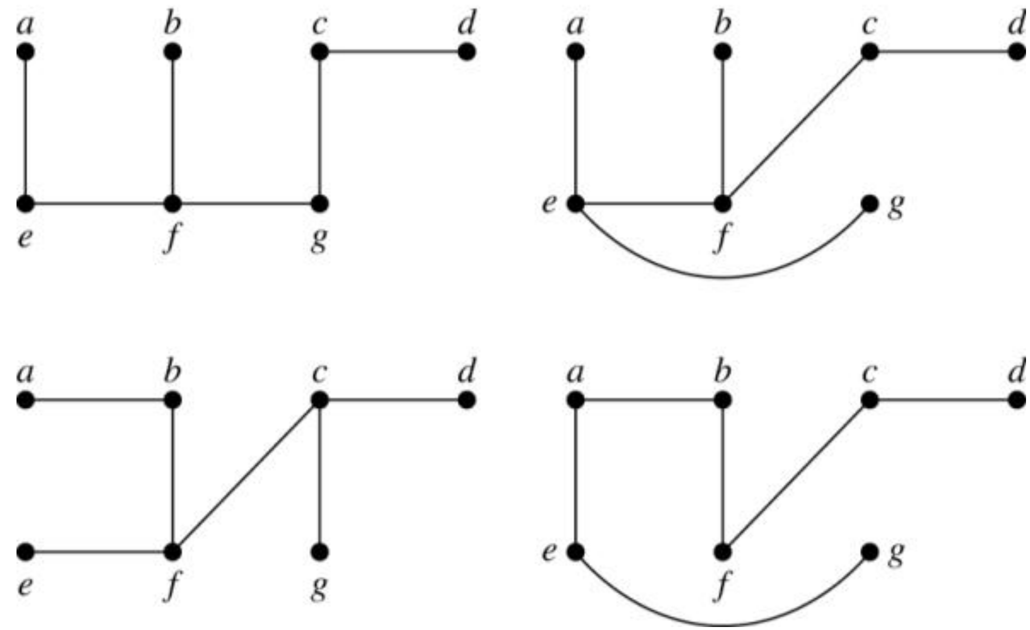
Remove an edge from any circuit.
(repeat until no circuit exists)



Spanning Trees



Four spanning trees of G:



Theorem: A simple graph is connected if and only if it has a spanning tree.

Depth-First Search (DFS)

Procedure *DFS*(G: connected graph with vertices $v_1, v_2, \ldots, v_n$)

T := tree consisting only of the vertex v_1

visit(v_1)

procedure *visit*(v: vertex of G)

for each vertex w adjacent to v and not yet in T

begin

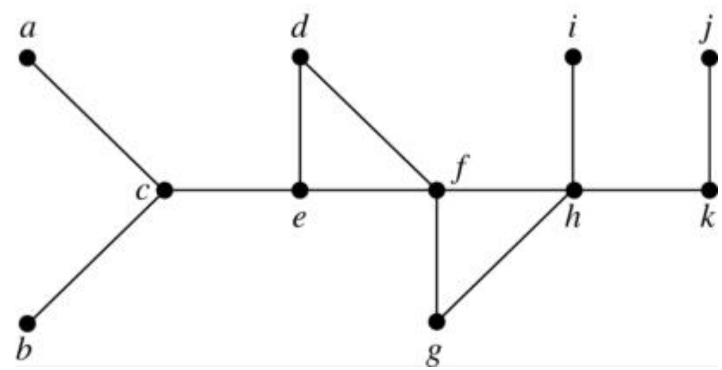
    add vertex w and edge $\{v, w\}$ to T

    visit(w)

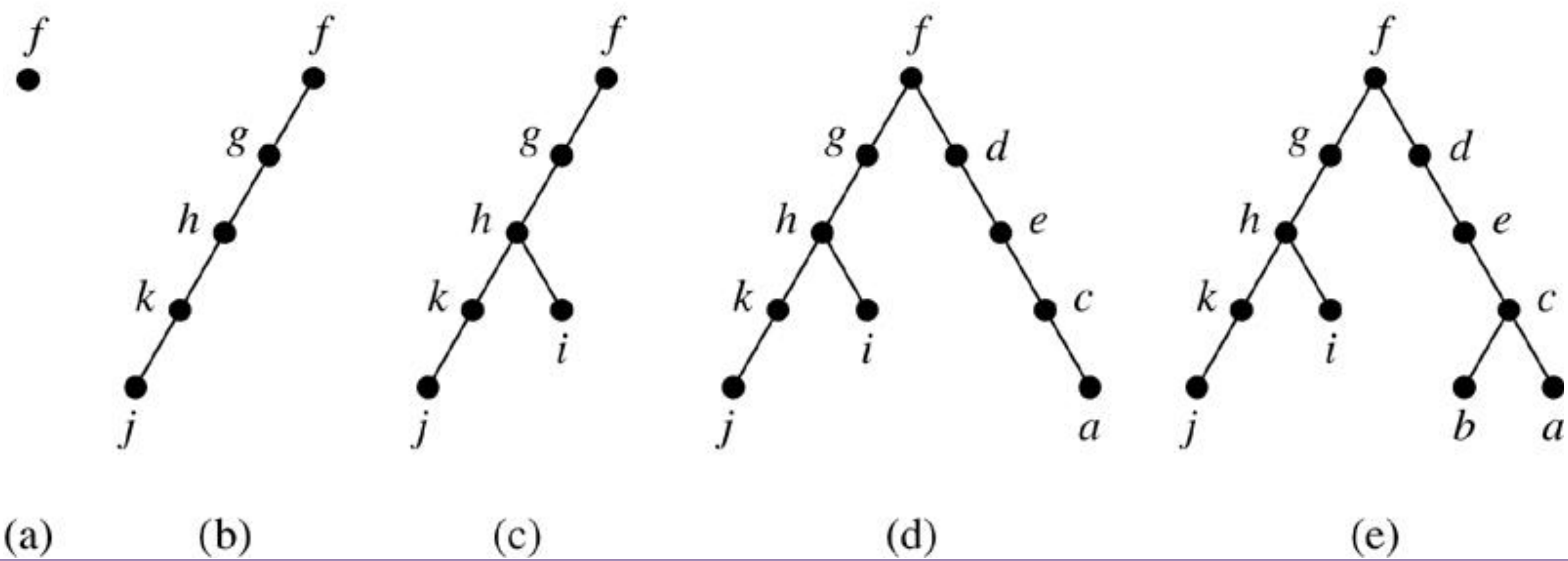
end

Example

- Use depth-first search to find a spanning tree for the graph.

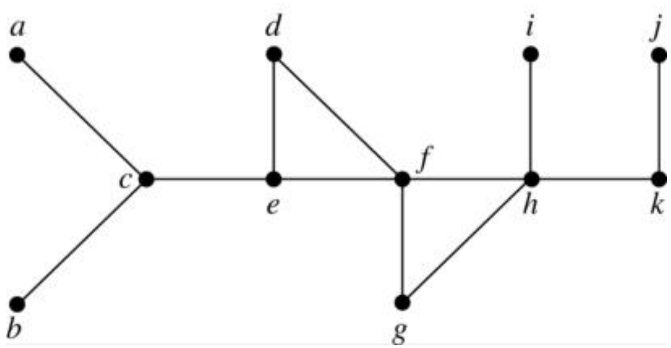


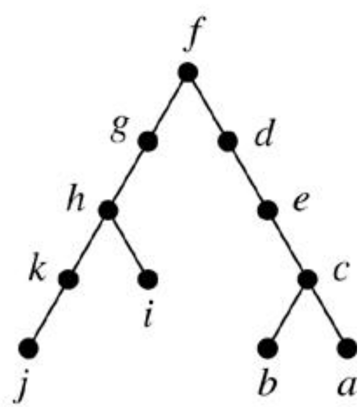
Solution. (arbitrarily start with the vertex *f*)



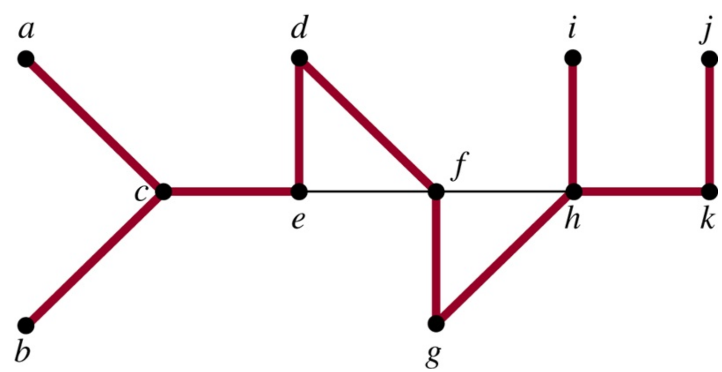
Example

- The edges selected by DFS of a graph are called tree edges. All other edges of the graph must connect a vertex to an ancestor or descendant of this vertex in the tree. These edges are called back edges.





$\Rightarrow$



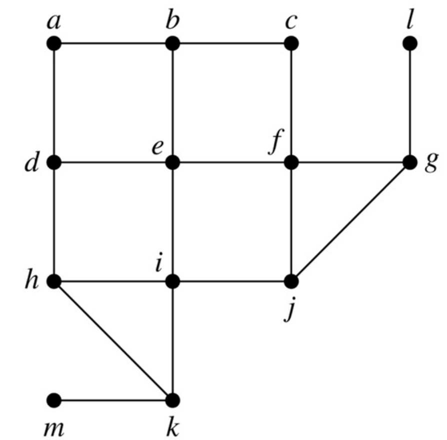
The tree edges (**red**)
and back edges (**black**)

Breadth-First Search (BFS)

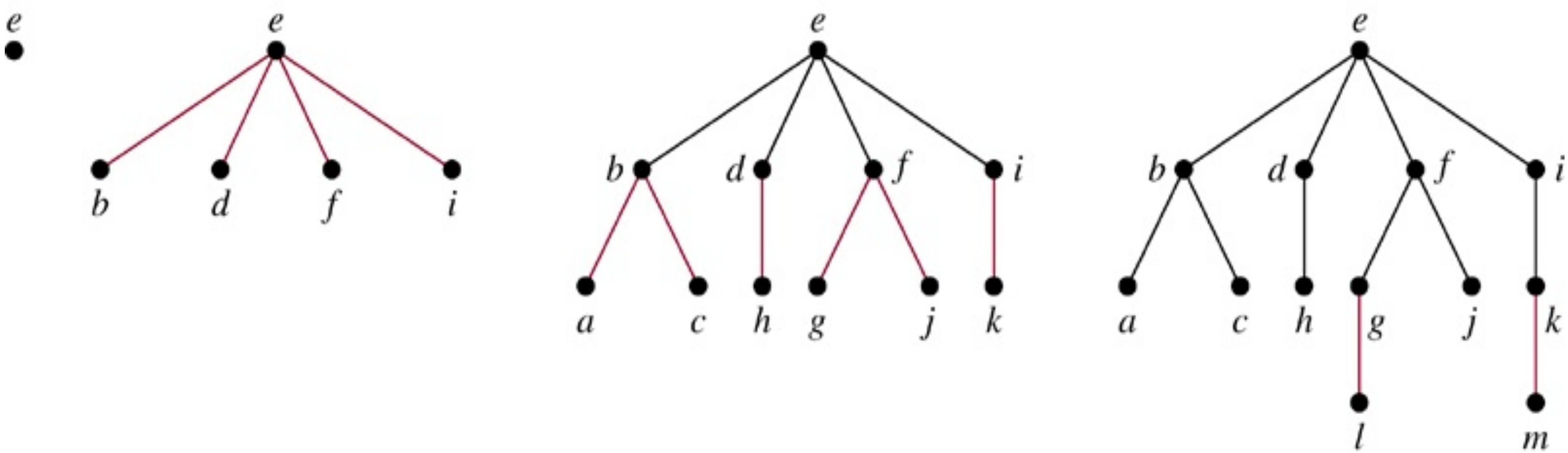
```
Procedure BFS(G: connected graph with vertices v1, v2, …, vn)
T := tree consisting only of vertex v1
L := empty list
put v1 in the list L of unprocessed vertices
while L is not empty
begin
     remove the first vertex v from L
     for each neighbor w of v
          if w is not in L and not in T then
          begin
               add w to the end of the list L
               add w and edge {v, w} to T
          end
end
```

Example

- Use breadth-first search to find a spanning tree for the graph



Solution: (arbitrarily start with the vertex e)



Minimum Spanning Trees

- A minimum spanning tree of G is a spanning tree with smallest sum of weights of its edges.
- Prim's Algorithm:

Procedure *Prim*(G: connected weighted undirected graph with n vertices)

T := a minimum-weight edge

for i := 1 **to** n–2

begin

    e := an edge of minimum weight incident to a vertex in T and not forming a simple circuit in T if added to T

    T := T with e added

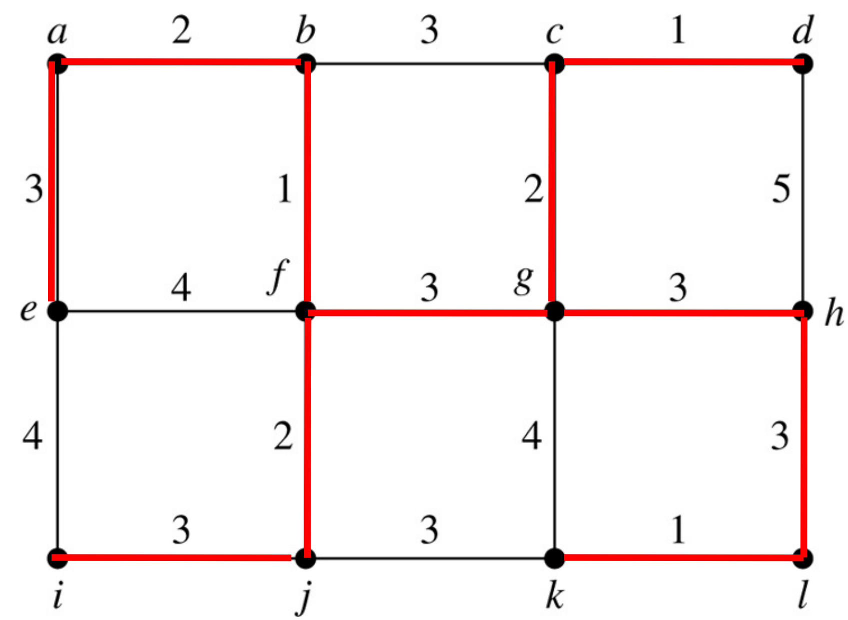
end {T is a minimum spanning tree of G}

Example

- Use Prim's algorithm to find a minimum spanning tree of G.

Solution:

Choice	Edge	Weight
1	$\{b, f\}$	1
2	$\{a, b\}$	2
3	$\{f, j\}$	2
4	$\{a, e\}$	3
5	$\{i, j\}$	3
6	$\{f, g\}$	3
7	$\{c, g\}$	2
8	$\{c, d\}$	1
9	$\{g, h\}$	3
10	$\{h, l\}$	3
11	$\{k, l\}$	1
	Total:	24



Kruskal Algorithm

Procedure *Kruskal*(G: connected weighted undirected graph with n vertices)

T := empty graph

for $i := 1$ **to** $n-1$

begin

  e := any edge in G with smallest weight that does not form a simple circuit when added to T

  $T := T$ with e added

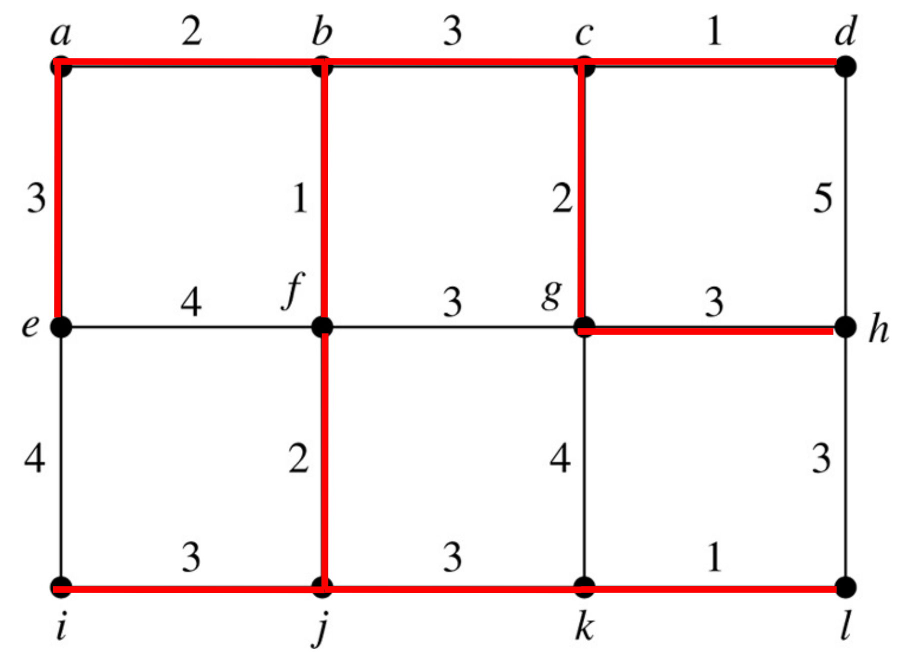
end {T is a minimum spanning tree of G}

Example

- Use Kruskal Algorithm to find a minimum spanning tree of G.

Solution:

Choice	Edge	Weight
1	$\{c, d\}$	1
2	$\{k, l\}$	1
3	$\{b, f\}$	1
4	$\{c, g\}$	2
5	$\{a, b\}$	2
6	$\{f, j\}$	2
7	$\{b, c\}$	3
8	$\{j, k\}$	3
9	$\{g, h\}$	3
10	$\{i, j\}$	3
11	$\{a, e\}$	3
	Total:	24



Next class

- Topic: Recap and Review (1)
- Pre-class reading: Chap 1-11